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Computational Geometric Learning

On the Complexity of Polyhedral Approximations of Submanifolds of Euclidean Spaces

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Introduction

In this report we examine some open questions with regard to the approximation of manifolds embedded in Euclidean space. Or discussion starts with a remark by Fejes Tóth in section 12 of chapter 5 of [5] that approximating a convex body with a polytope, where we allow the vertices to lie in the ambient space, is twice as efficient as approximating the same convex body with a polytope, whose vertices lie on the surface, if the number of vertices tends to infinity. In section 1 we provide a proof for this statement. Secondly we prove that the surface area of an optimal triangulation of a two dimensional compact surface embedded in Euclidean space has a uniform bound, which provided an inspiration for further work. We also provide place the following remark, for a family of triangulations T_n, having n vertices, of a surface Σ whose Hausdorff-distance decreases fast, that is $d^H(T_n, \Sigma) \sim 1/n$ it is not clear whether the area of T_n tends to the area of Σ. This is a generalization of the famous lantern of Schwarz [17], see also [13], in the sense that both $d^H(L_n, \Sigma)$ and $d^H(T_n, \Sigma)$ tend to zero as n tends to infinity, but $d^H(L_n, \Sigma)$ does so much slower then $1/n$, where L_n denotes the lantern with n vertices. Finally, in the third main section, we turn our attention to ruled surfaces embedded in three dimensional Euclidean space, which are special in the following sense; because straight lines are contained in the surface it is no longer clear whether the edge-length of the triangles in an optimal triangulation tends to zero as the Hausdorff distance decreases to zero. Fejes Tóth [5] in fact believed that it was possible to use such triangles to construct a very good approximation of the one-sheeted hyperboloid. This is unfortunately incorrect. We shall state the following conjecture:

Let Σ be a compact manifold of nonzero Gaussian curvature, possibly with a boundary and let T_n be the sequence of optimal triangulations with n vertices, where each T_n is assumed to be isotopic to Σ. Then the length of each edge of every triangle in the sequence t^n, where $t^n \in T_n$ tends to zero as n tends to infinity.

This conjecture is verified in some special cases furthermore there is strong numerical evidence that the statement is correct. The numerical work shall not be presented here. This conjecture implies one of the first statements of the first section because in this case the normals of the triangulation converge to the normals of the surface so that the areas of both with converge as well, compare with Hildebrandt et al. [11].

We have appended a discussion of the approximation of C^2 convex surfaces by C^3 (or C^∞) convex surfaces. This might be considered of interest because in [8] and [9] Gruber goes to great length to generalize the results of Schneider [16] to the class of C^2 convex surfaces. We later discovered that this idea has already been explored by Greene and Wu [7]. Because the result is so short and is exposed in a manner which is immediately applicable we have included it.

The outline of this report is such that each individual section can be examined separately.

1 On the optimal triangulation of convex hypersurfaces, whose vertices lie in ambient space.

We consider triangulations of surfaces. The purpose of this section is firstly to make apparent that if we allow vertices to lie in the ambient space of a surface we are able to reduce the Hausdorff distance between the surface and its approximation via simplicial complices. To illustrate this general idea we shall consider the d-sphere S^d with the standard embedding in $\mathbb{R}^{d+1}$. Moreover we relate the optimal Hausdorff distance of an inscribed polytope with a given number of vertices to the optimal Hausdorff distance of a simplicial complex those vertices are in a more general position.

We start with the circle S^1 in $\mathbb{R}^2$. We assume that the circle has radius 1 and its center lies in the origin and we approximate the circle by a regular polygon, with n vertices P_n. Due to symmetry this is the obvious and optimal manner. Naturally the center of the regular polygon is the origin. The distance between the vertices of the regular polygon and the origin will be denoted by R. The situation is sketched in figure 1. There are clearly two kinds of points of interest on the edges of the polygon, namely the vertices and the center of the edge. Independent of R these are the points which are either the furthest way from or closest to the circle. This is easy to see due to the fact that for a point $p \in \mathbb{R}^2 \setminus (\{0\} \cup S^1)$ the point on the circle S^1 which is closest is p is the point of the circle which lies on the halfline which originates in the origin and intersects p. Furthermore it is obvious that the distance between the vertex and the circle is given by $1 - R$ and the distance between the center of the edge and the circle by $1 - R\cos(2\pi/n)$, which reads in the inscribed case $1 - \cos(2\pi/n)$. From these considerations we find that the Hausdorff distance between the circle and the regular polygon is given by

$$d^H(P_n, S^1) = \max\left\{R - 1, 1 - R(1 - d^H(P_n^{\text{in}}, S^1))\right\} = \max\left\{R - 1, 1 - R\cos\left(\frac{2\pi}{n}\right)\right\},$$

where P_n^{in} denotes the inscribed polygon that is the polygon with $R = 1$.

We can minimize $d^H(P_n, S^1)$ with respect to R by the following choice

$$R = \frac{2}{1 + \cos\left(\frac{2\pi}{n}\right)} = \frac{2}{2 - d^H(P_n^{\text{in}}, S^1)}.$$

So that

$$d^H(P_n, S^1) = \frac{2}{1 + \cos\left(\frac{2\pi}{n}\right)} - 1 = \frac{2}{2 - d^H(P_n^{\text{in}}, S^1)} - 1,$$

for sufficiently large n or rather sufficiently low $d^H(P_n^{\text{in}}, S^1)$ we can develop this expression

$$d^H(P_n, S^1) = \frac{2}{1 + \cos\left(\frac{2\pi}{n}\right)} - 1 \simeq \frac{1}{4}\left(\frac{2\pi}{n}\right)^2 + \mathcal{O}\left(\left(\frac{2\pi}{n}\right)^4\right)$$

$$d^H(P_n, S^1) = \frac{2}{2 - d^H(P_n^{\text{in}}, S^1)} - 1 \simeq \frac{1}{2} d^H(P_n^{\text{in}}, S^1) + \mathcal{O}((d^H(P_n^{\text{in}}, S^1)^2).$$

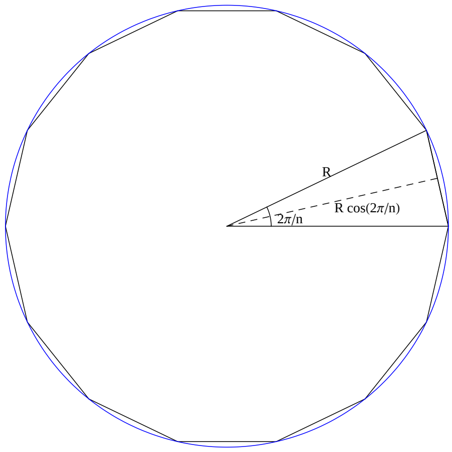

Figure 1: A polygon and a circle, both with the same center. We have chosen to depict the inscribed case.

We shall now generalize this discussion to general spheres denoted by S^d. To do so we shall generalize the remark that for a point $p \in \mathbb{R}^{(d+1)} \setminus (\{0\} \cup S^d)$ the point on S^d which is closest is p is the point of the circle which lies on the halfline which originates in the origin and intersects p, to arbitrary S^d. We shall give the proof in a lemma for $S^2 \subset \mathbb{R}^3$, the proof generalizes trivially to any given d.

Lemma 1 *For a point $p \in \mathbb{R}^3 \setminus (\{0\} \cup S^2)$ the point on S^2 which is closest is p is the point which lies on the halfline which originates in the origin and intersects p.*

Proof We start by picking a parametrization of the sphere namely

$$s(\phi, \theta) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta).$$

Due to rotational symmetry of the problem we may assume that $p = (r, 0, 0)$, with $r \in (0, 1) \cup (1, \infty)$. The distance of the point p to a point on the sphere with coordinates θ and ϕ is given by

$$d(s(\phi, \theta), p) = \sqrt{1 + r^2 - 2r \cos\phi \sin\theta}$$

taking the gradient yields that critical points are given by

$$(\phi, \theta) \in (\pi\mathbb{Z}, \pi/2 + \pi\mathbb{Z}) \cup (\pi/2 + \pi\mathbb{Z}, \pi\mathbb{Z}),$$

corresponding to the points $(\pm 1, 0, 0)$ and $(0, 0, \pm 1)$ on the sphere. Direct calculation shows that the minimum is attained in $(1, 0, 0)$. Clearly $(1, 0, 0)$ is the point which lies on the halfline which originates in the origin and intersects p. □

For general S^d we are not able to give a concrete description of the inscribed polytope which approximates S^d optimally, however suppose that we are give such a polytope P_n^{in}, then the vertices of P_n^{in} lie on S^d and there are points on the faces which attain a distance $d^H(P_n^{\mathrm{in}}, S^d)$ to the sphere. We can now consider the polytope P_n^{in} rescaled by a factor R, denoted by $R\,P_n^{in}$. For $R\,P_n^{\mathrm{in}}$, the afore mentioned points, the vertices and the points where a the maximum distance is attained, are of special interest because independent of R these are the points which are either the furthest way from or closest to S^d. This means that

$$d^H(R\,P_n^{in}, S^d) = \max\left\{R-1, 1-R(1-d^H(P_n^{\mathrm{in}}, S^d))\right\},$$

which is minimized with respect to R by

$$R = \frac{2}{2-d^H(P_n^{\mathrm{in}}, S^d)}.$$

So that

$$d^H(R\,P_n^{in}, S^d) = \frac{2}{2-d^H(P_n^{\mathrm{in}}, S^d)} - 1 \simeq \frac{1}{2}d^H(P_n^{\mathrm{in}}, S^d) + \mathcal{O}((d^H(P_n^{\mathrm{in}}, S^d)^2).$$

The only thing left to proof is that there is indeed an inscribed polytope P_n^{in} which approximates S^d optimally. We shall give this proof in the form of a lemma.

Lemma 2 *For each n there exists an inscribed polytope P_n^{in} which approximates a given compact convex surface C optimally. Here we specifically allow the vertices to coincide.*

Proof It suffices to show that $d^H(P_n^{\mathrm{in}}, C)$, where P_n^{in} denotes a polytope with n vertices, is continuous with respect to the vertices. This is due to the fact that each of the vertices is chosen on C and therefore $d^H(P_n^{\mathrm{in}}, C)$ may be seen as a function from C^n to $\mathbb{R}$. Furthermore C and therefore C^n is compact and a continuous function attains its minimum on a compact set.
Continuity however is a direct consequence of the triangle inequality. Let $v_1, v_2 \in C^n$ be such that $|v_1 - v_2| \leq \delta$. The triangle inequality yields

$$d^H(v_2, C) - d^H(v_1, v_2) \leq d^H(v_1, C) \leq d^H(v_2, C) + d^H(v_1, v_2),$$

where for convenience we identified a simplex with its vertices. Clearly $d^H(v_1, v_2) \leq 4\delta$, because if the vertices of two simplices are close the simplices are close in the Hausdorff sense due to linearity. This implies that for $|v_1 - v_2| \leq \delta$ we have $|d^H(v_2, C) - d^H(v_1, C)| \leq 4\delta$ and therefore $d^H(\cdot, C) : C^n \to \mathbb{R}$ is continuous. □

This lemma will also be the first step toward an extension of the discussion of the sphere to more general convex surfaces. In the following lemma we mimic lemma 2 to obtain a similar result for triangulations whose vertices do not lie on the surface.

Lemma 3 *Suppose that we have $q \in \mathbb{R}^{d+1}$, $r \in \mathbb{R}$ and a polytope P_m , such that $C \subset B(q, r)$, $d^H(C, \partial B(q, r)) \leq \Delta$ and $d^H(C, P_m) < \Delta$, for some $\Delta \in \mathbb{R}$, where $B(q, r)$ indicates the ball with center q and radius r. Then there exists a polytope P_n for each $n \geq m$, which approximates C optimally.*

Proof As in lemma 2, we may consider $d^H(P_n, C)$ to be a continuous function from $(B(q,r))^n$ to $\mathbb{R}$. Naturally it attains its minimum on the compact set $(B(q,r))^n$. This minimum must be global because if there would be an optimal polytope $\tilde{P}_n$ such that $\tilde{P}_n \not\subseteq B(r,q)\backslash\partial B(r,q)$, then $d^H(\tilde{P}_n, C) \geq \Delta$, which contradicts $d^H(\tilde{P}_n, C) \leq d^H(C, P_m) < \Delta$. □

Remark 4 *In the lemma above we use the obvious statement that for two optimal simplicial complices (or polytopes) S_n and S_m, where $n < m$ we have that $d^H(S_n, C) \leq d^H(S_m, C)$. However a strict inequality does not hold in the most general setting. An example of this is the following; consider the circle C and its optimal approximating simplicial complices for $n = 1$ and $n = 2$, that is a dot in the center and a line segment which intersects the center and does not extend beyond twice the radius of the circle. It is easy to see that $d^H(S_1, C) = d^H(S_2, C)$.*

The following lemmas discusses the relation between the Hausdorff distance of triangulations whose vertices are restricted to the surface and those who are not restricted to a given convex surface.

Lemma 5 *For a given n suppose that we have an optimally approximating simplicial complex S_n (in general not inscribed), that is there is no $\tilde{S}_n$ such that $d^H(\tilde{S}_n, C) < d^H(S_n, C)$ then the optimally approximating inscribed polytope P_n with n vertices satisfies $d^H(P_n, C) \leq 2d^H(S_n, C)$*

Proof For each vertex v_i of S_n consider a point v_i^{on} on C closest to v_i. Then under the mapping $v_{i_0} \dots v_{i_n} \mapsto v_{i_0}^{\text{on}} \dots v_{i_n}^{\text{on}}$, where again we identify a simplex with its vertices, we find a simplicial complex S_n^{on}. Due to linearity we have that for each simplex $d^H(v_{i_1} \dots v_{i_n}, v_{i_1}^{\text{on}} \dots v_{i_n}^{\text{on}}) \leq d^H(S_n, C)$ and therefore $d^H(S_n, S_n^{\text{on}}) \leq d^H(S_n, C)$, which in turn using the triangle inequality yields

$$d^H(S_n^{\text{on}}, C) \leq d^H(S_n^{\text{on}}, S_n) + d^H(S_n, C) \leq 2d^H(S_n, C)$$

by definition of optimality on enclosed polytopes we have that $d^H(P_n, C) \leq d^H(S^{\text{on}}, C)$ which yields that $d^H(P_n, C) \leq 2d^H(S_n, C)$ □

For the following lemma we have need of a result Schneider [16], in particular the second lemma (Hilfssatz). This lemma relates the covering of the surface by II-geodesic balls to the number of vertices which is needed to attain a certain Hausdorff distance of an inscribed polytope to the surface. Moreover its proof discusses the placement of the vertices. From the proof we may conclude that the length of the edges of an optimally approximating polytope tends to zero as the Hausdorff distance between the inscribed polytope and surface tends to zero. In the following lemma we also use the reach of a manifold, which is defined to be

Definition 6 *Let M be an $n-1$-dimensional manifold embedded in $\mathbb{R}^n$ and let $\pi : \mathbb{R}^n \to M$ be a (possibly) multi-valued map which maps each point of $\mathbb{R}^n$ to its closest points on M. The reach of M, $R(M)$, is defined as the supremum of all δ such that any point of $\mathbb{R}^d$ lying at distance less then δ from M has a unique image.*

In this definition we followed Boissonat and Gosh [1]. It has been shown by Federer [4] that a C^2 manifold has strictly positive reach. We are now ready to state the following lemma.

Lemma 7 *Let P_n^{on} be an optimally approximating polytope with Hausdorff distance $d^H(P_n^{on}, C)$ to the surface. Where we assume that the reach of $d^H(P_n^{on}, C)$ is smaller then reach of C. Then we can construct a polytope $\tilde{P}_n$ such that*

$$d^H(\tilde{P}_n, C) \leq \frac{1}{2} d^H(P_n^{on}, C) + o(d^H(P_n^{on}, C)).$$

Proof Let v_i be the vertices of P_n^{on} then we choose the vertices $\tilde{v}_i$ of our $\tilde{P}_n$ to be $v_i + \frac{1}{2}\nu(v_i) d^H(P_n^{\text{on}}, C)$, where ν denotes the normal to the surface. Naturally $\tilde{P}_n$ is the convex hull of all the vertices $\tilde{v}_i$. To see why this cuts the Hausdorff distance in half, up to higher order, we turn to the alternative definition of the Hausdorff distance, see Munkres [14]:

$$d^H(A, B) = \inf\{\epsilon | A \subset U(B, \epsilon) \text{ and } B \subset U(A, \epsilon)\},$$

where $U(A, \epsilon)$ denotes the ϵ-neighbourhood of A. From this we see that P_n^{on} is contained in an inner rim inside the convex surface $C = \partial M$ that is

$$P_n^{\text{on}} \subset U(C, d^H(P_n^{\text{on}}, C)) \cap M.$$

We also have that

$$C \subset U(P_n^{\text{on}}, d^H(P_n^{\text{on}}, C)).$$

This yields that for every point $x \in P_n^{\text{on}}$ we have a unique point $y \in C$ which is closest to x, moreover the vector $(x - y)$ is normal to the surface. We may now define the mapping N by

$$N : x \mapsto x + \frac{1}{2} d^H(P_n^{\text{on}}, C) \frac{x - y}{|x - y|}$$

and consider $N(P_n^{\text{on}})$. By definition we have that

$$N(P_n^{\text{on}}) \subset U(C, 1/2 d^H(P_n^{\text{on}}, C)).$$

We shall now show that

$$C \subset U(N(P_n^{\text{on}}), \frac{1}{2} d^H(P_n^{\text{on}}, C)).$$

This shall be made apparent as the follows. For every $y \in C$ there is a $x \in P_n^{\text{on}}$ such that $y - x$ is normal to C and $|y - x| \leq d^H(P_n^{\text{on}}, C)$. The first intersection point of $\{c - \lambda\nu(x) | \lambda \in \mathbb{R}\}$ and P_n^{on} will do, where by the first we mean the point with the smallest λ associated. There is such an intersection point due to the following; suppose there exists a $y \in C$ such that

$$\{y - \lambda\nu(y) | \lambda \in \mathbb{R}\} \cap P_n^{\text{on}} = \emptyset,$$

then the line $\{y - \lambda\nu(y) | \lambda \in \mathbb{R}\}$ intersects C at some other point $\tilde{y} = y - \tilde{\lambda}\nu(y)$, without first intersecting P_n^{on}. This also means that there is a point $y_e = y - \tilde{\lambda}\nu(y)/2$, which has equal distance to y and $\tilde{y}$ so y_e lies further from C then the reach but this contradicts the assumption that $d^H(P_n^{\text{on}}, C) \leq R(C)$. We will now show that $|y - x| \leq d^H(P_n^{\text{on}}, C)$. Suppose that there is a y' such that $|y' - x| \leq |y - x|$, then we can find a point along $y - x$ with equal distance to two points of C, namely y and y', again contradicting the assumption that $d^H(P_n^{\text{on}}, C) \leq R(C)$. Therefore y is the point on C which is closest to x and thus $|y - x| \leq d^H(P_n^{\text{on}}, C)$. Given the special role we have thrust on the normal ν it is clear that

$$C \subset U(N(P_n^{\text{on}}), \frac{1}{2} d^H(P_n^{\text{on}}, C)).$$

This implies that

$$d^H(N(P_n^{\text{on}}), C) \leq \frac{1}{2} d^H(P_n^{\text{on}}, C).$$

We now use that $d^H(N(P_n^{\text{on}}), \tilde{P}_n)$ tends to zero as n tends to infinity, faster then $d^H(P_n^{\text{on}}, C)$ tends to zero. This holds because the surface is smooth and the length of the edges of each triangle T tends to zero so that the normals $(x-y)/|x-y|$, where x is an element of T, line up with $\nu(v_i)$, where v_i is a vertex of T. This implies that $(x-y)/|x-y| - v_i$ tends to zero so $d^H(P_n^{\text{on}}, C)((x-y)/|x-y| - \nu(v_i))$, which gives an upper bound on $d^H(N(P_n^{\text{on}}), \tilde{P}_n)$, tends to zero faster then $d^H(P_n^{\text{on}}, C)$. So we have that

$$d^H(\tilde{P}_n, C) \leq d^H(N(P_n^{\text{on}}), \tilde{P}_n) + d^H(N(P_n^{\text{on}}), C) \leq \frac{1}{2} d^H(P_n^{\text{on}}, C) + o(d^H(P_n^{\text{on}}, C)).$$

So $\tilde{P}_n$ is a simplicial complex sufficiently close to C. □

Lemmas 2, 3, 5 and 7 have the following corollary:

Corollary 8 *For every n let S_n be an optimally approximating simplicial complex with n vertices having Hausdorff distance $d^H(S_n, C)$ to the surface. Then we have*

$$\lim_{n\to\infty} n^{2/(d-1)} d^H(S_n, C) = \frac{1}{4} \left(\frac{\theta_{d-1}}{k_{d-1}} \int_C \sqrt{K_C(x)} \mathrm{d}\mu \right)^{2/(d-1)},$$

where k_d is the volume of the d-dimensional ball $\pi^{d/2}/\Gamma(1 + d/2)$, θ_d is the covering density of the ball in d-dimensional space and K the Gaussian curvature.

Proof By Gruber and Schneider [8, 16] we have that

$$\lim_{n\to\infty} n^{2/(d-1)} d^H(P_n^{\text{on}}, C) = \frac{1}{2} \left(\frac{\theta_{d-1}}{k_{d-1}} \int_C \sqrt{K_C(x)} \mathrm{d}\mu \right)^{2/(d-1)}.$$

Lemmas 5 and 7 give us

$$\frac{1}{2} d^H(P_n, C) \leq d^H(S_n, C)$$

and

$$d^H(\tilde{P}_n, C) \leq \frac{1}{2} d^H(P_n^{\text{on}}, C) + o(d^H(P_n^{\text{on}}, C)).$$

And lemmas 3 and 5 give us the existence of the simplicial complices involved. By optimality we the latter equation implies

$$d^H(S_n, C) \leq \frac{1}{2} d^H(P_n^{\text{on}}, C) + o(d^H(P_n^{\text{on}}, C)).$$

So that

$$\begin{aligned} \frac{1}{2} \lim_{n\to\infty} n^{2/(d-1)} d^H(P_n^{\text{on}}, C) \leq \lim_{n\to\infty} n^{2/(d-1)} d^H(S_n, C) &\leq \lim_{n\to\infty} n^{2/(d-1)} \left(\frac{1}{2} d^H(P_n^{\text{on}}, C) + o(d^H(P_n^{\text{on}}, C)) \right) \\ &= \frac{1}{2} \lim_{n\to\infty} n^{2/(d-1)} d^H(P_n^{\text{on}}, C). \end{aligned}$$

Using the result of Gruber and Schneider yields

$$\lim_{n\to\infty} n^{2/(d-1)} d^H(S_n, C) = \frac{1}{4} \left(\frac{\theta_{d-1}}{k_{d-1}} \int_C \sqrt{K_C(x)} \mathrm{d}\mu \right)^{2/(d-1)},$$

the desired result. □

2 On the surface area of two dimensional surfaces and their optimal triangulation.

The purpose of this section is to prove that the surface area of an optimal triangulation of a 2-dimensional compact surface embedded in Euclidean space has a uniform bound, under the assumption that the triangulation is itself a topological 2-manifold.
We use the expression optimal triangulation in the sense of the following definition:

Definition 1 *For every $\epsilon > 0$ let $n(\Sigma, \epsilon)$ be the smallest number n such that there is a triangulation T_n with n vertices for which $d^H(\Sigma, T_n) \leq \epsilon$ holds.*

We shall use the following results:

1. If n is the number of vertices in a triangulation (which is itself a topological 2-manifold) then the triangulation contains at most $\mathcal{O}(n)$ simplices.

2. There exists a family of triangulations S_n, having n vertices and being topological 2-manifolds, of the surface with the following properties:

$$\max_j \|\mathrm{E}_j(S_n)\| = \Theta(1/n^{1/2})$$
$$d^H(S_n, \Sigma) = \Theta(1/n)$$
$$\lim_{n\to\infty} \mathrm{A}(S_n) = \mathrm{A}(\Sigma), \tag{1}$$

   where $\mathrm{E}_j(S_n)$ indicates the edge of the simplicial complex and $\mathrm{A}(M)$ indicates the area of M.

We can now use the family of triangulations S_n to prove the following lemma

Lemma 2 *Let Σ be a 2-dimensional compact surface embedded in Euclidean space. And let T_n be a family of triangulations with n vertices such that $d^H(\Sigma, T_n) \leq d^H(\Sigma, S_n)$, T_n is a topological 2-manifold for every n as well as the limit T_n for n tends to infinity. Then the surface area (volume) of T_n is bounded.*

Proof
We start by considering the following simple case; suppose a polygon $\tilde{p}$ is embedded in $\mathbb{R}^3$ and lies within distance ϵ from the xy-plane. Then the difference in surface area of $\tilde{p}$ and its projection onto the xy-plane is clearly bounded by

$$4|\partial\tilde{p}|\epsilon,$$

where $\partial\tilde{p}$ denotes the perimeter of the triangle t_n and $|\partial\tilde{p}|$ its length. We now focus on an individual triangle in S_n, which we denote by s_n. The open neighbourhood U_{s_n} of s_n of radius ρ may be defined as follows

$$U_{s_n} = \bigcup_{x \in s_n} B(x; \rho).$$

Form this point onward we assume that $\rho = 2d^H(\Sigma, S_n)$. Note that $d^H(S_n, T_n) \leq (d^H(\Sigma, T_n) + d^H(\Sigma, S_n) \leq 2d^H(\Sigma, S_n)$ due to the triangle inequality. This enables us to compare the surface area of the triangulation in this neighbourhood $T_n \cap U_{s_n}$, with the surface area of the triangle S_n. We start by projecting the edges within $T_n \cap U_{s_n}$ onto s_n. The edges of T_n within U_{s_n} will be denoted by $\partial(T_n) \cap U_{s_n}$. In figure 2 a sketch of the situation is given.

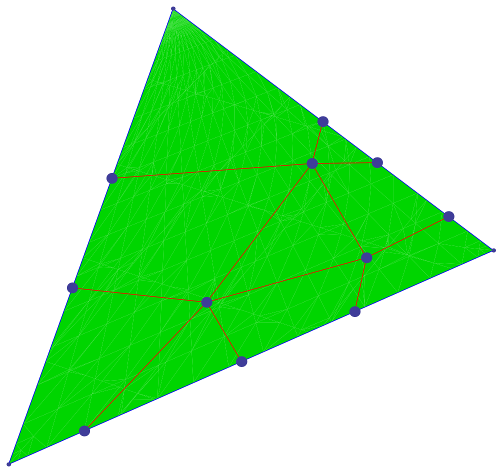

Figure 2: The projection of the perimeters of the triangles t_n on the triangle s_n (indicated in red) as well as the perimeter of the s_n (indicated in blue).

From the simple case discussed before we can conclude that the difference in surface area between $T_n \cap U_{s_n}$ and s_n is bounded by

$$\begin{aligned}
&4|\partial(T_n) \cap U_{s_n}|d^H(S_n, T_n) + 4|\partial s_n|d^H(S_n, T_n) \\
&\leq 4|\partial(T_n) \cap U_{s_n}|(d^H(\Sigma, T_n) + d^H(\Sigma, S_n)) + 4|\partial s_n|(d^H(\Sigma, T_n) + d^H(\Sigma, S_n)) \\
&\leq 4|\partial(T_n) \cap U_{s_n}|2d^H(\Sigma, S_n) + 4|\partial s_n|2d^H(\Sigma, S_n) \\
&= 8(|\partial(T_n) \cap U_{s_n}| + |\partial s_n|)d^H(\Sigma, S_n),
\end{aligned}$$

where again we have applied the triangle inequality to the Hausdorff distance. We also have need of the results quoted above, in particular we shall use that there are c_1, c_2, c_3, c_4 such that

$$
\begin{aligned}
N(S_n) &\leq c_1 n \\
\max_j \|\mathrm{E}_j(S_n)\| &\leq c_2/n^{1/2} \\
d^H(S_n, \Sigma) &\leq c_3/n, \\
N(T_n) &\leq c_4 n
\end{aligned}
$$

where $N(S_n)$ denotes the number of simplices of the complex S_n. We implicitly use that T_n is a topological 2-manifold. Finally we note that because Σ is compact the surface is bounded and therefore the area of each triangle in a triangulation is bounded as well. We can now proceed to the determination of the difference between the surface area's (denoted by A) of S_n and T_n

$$
\begin{aligned}
|\mathrm{A}(S_n) - \mathrm{A}(T_n)| &\leq \sum_{s_n \in S_n} 8(|\partial(T_n) \cap U_{s_n}| + |\partial s_n|) d^H(\Sigma, S_n) \\
&\leq \sum_{s_n \in S_n} 8|\partial(T_n) \cap U_{s_n}| d^H(\Sigma, S_n) + \sum_{n \leq N(S_n)} 8|\partial s_n| d^H(\Sigma, S_n) \\
&\leq 16|\partial(T_n)| d^H(\Sigma, S_n) + 24 c_1 n c_2/n^{1/2} d^H(\Sigma, S_n) \\
&\leq 48 c_4 n d^H(\Sigma, S_n) + 24 c_1 n c_2/n^{1/2} d^H(\Sigma, S_n) \\
&\leq 48 c_4 c_3 n/n + 24 c_1 c_2 c_3 (n/n) 1/n^{1/2} \\
&= 48 c_4 c_3 + 24 c_1 c_2 c_3/n^{1/2}.
\end{aligned}
$$

Since Σ is compact and thus has a finite volume and $\mathrm{A}(S_n)$ converges to $\mathrm{A}(\Sigma)$ we can conclude that $\lim \mathrm{A}(T_n) < \infty$. □

Given this result we must also give the following warning:

Remark 3 *Let Σ be a 2-dimensional compact surface embedded in Euclidean space. And let T_n be a family of triangulations with n vertices such that $d^H(\Sigma, T_n) \leq d^H(\Sigma, S_n)$, where where S_n is a given family of triangulations satisfying (1). Then it is not necessarily true that*

$$\lim_{n\to\infty} A(T_n) = A(\Sigma).$$

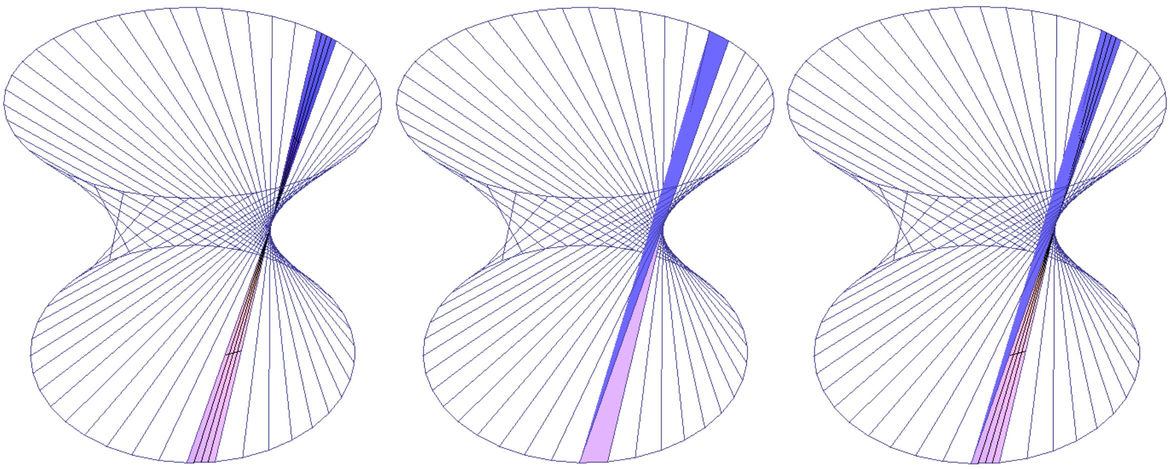

Figure 3: In each picture one set of rulings of the hyperboloid is depicted, in the first we also depict a small part of hyperboloid, in the second two triangles one of each type, in the third we have combined both.

To show that this is not the case we turn our attention to the one-sheeted hyperboloid, for convenience with boundary. More explicitly, we take the hyperboloid to be given by

$$\{f(\theta, l) | \theta \in [0, 2\pi], l \in [-2, 2]\},$$

where

$$f(\theta, l) = (\cos\theta - l\sin\theta, \sin\theta + l\cos\theta, l)$$

the triangulation T_{2n}, with $2n$ vertices, we consider consists of two types of triangle namely

$$\triangle\, f\left(\frac{2\pi i}{n}, 2\right) f\left(\frac{2\pi i}{n}, -2\right) f\left(\frac{2\pi(i+1)}{n}, 2\right)$$

and

$$\triangle\, f\left(\frac{2\pi(i+1)}{n}, -2\right) f\left(\frac{2\pi(i+1)}{n}, 2\right) f\left(\frac{2\pi i}{n}, -2\right),$$

where $0 \leq i \leq n$. Although the exact Hausdorff distance of the triangulation to the surface is difficult to determine we are able to give the following estimation of the upper bound:

$$d^H(\Sigma, T_{2n}) \lesssim 1 - (\cos(\pi/n) - 2\sin(\pi/n)).$$

This means that the Hausdorff distance of the triangulation is fairly good while the number of vertices is quite low. Moreover we can easily calculate the surface of the hyperboloid

$$(12 + \sqrt{12}\text{arcsinh}(2\sqrt{2}))\pi$$

and the triangulation T_{2n}

$$4\sqrt{2}n\sqrt{\sin^2\frac{\pi}{n}\left(15 + 3\cos\frac{2\pi}{n} + 4\sin\frac{2\pi}{n}\right)},$$

which converges to 24π as n tends to infinity. So we find that although there is clearly a S_{2n} which satisfies (1) as well as conforming to $d^H(\Sigma, T_{2n}) \leq d^H(\Sigma, S_{2n})$, we do not get convergence of the area.

3 On the asymptotic behaviour of triangles in an optimal triangulation

Introduction

Our main object is to find an optimal triangulation of a surface in the sense of Schneider [16], that is we use as few points as possible to achieve good accuracy in sense of the Hausdorff distance. We restrict ourselves to the asymptotic setting, where the number of vertices is large. Schneider [16] and Gruber [8] have shown in the convex case that these optimal solutions are strongly related to point samples that are uniformly distributed on the surface with respect to the metric induced by the second fundamental form. Work on the generalization of this to non-convex surfaces has already been published by Clarkson [2], but this contains no definite answer with regard to the lower bound on the Hausdorff distance. One of the major problems one faces when considering curved surfaces of negative Gaussian curvature is that the edge length is no longer guaranteed to decrease if the Hausdorff distance between the surface and its triangulation decreases. More specifically a surface of negative curvature may contain straight lines. For individual lines this is of no concern, the set of all straight lines has measure zero and adding more points on this set then strictly necessary, to decrease the edge length, will not significantly influence the development of the Hausdorff distance in terms of the number of vertices, that is the influence of these points shall not be leading. If the straight lines of the surface lie densely on the surface as is the case with a ruled surface it is no longer clear that the possibility of the presence of triangles with long edges has little influence. This is the topic which we shall investigate further in the following text.

Ruled surfaces

In this subsection we discuss some differential geometric properties of ruled surfaces. We review must needed classical results and derive a corollary that relates the Gaussian curvature of a ruled surface to the behaviour of the normal along a fixed ruling. This result gives some geometrical insight into the special role of ruled surfaces in triangulations

A C^1 surface Σ embedded in $\mathbb{R}^3$ is called ruled if for every point of Σ there exists a (straight) line (segment) $L_p \subset \Sigma$ with $p \in L_p$. If we now consider a small neighborhood U in Σ of a given point p_0 on some line (segment) L we choose a C^1 curve p_t on Σ whose tangent line is not pointing in the direction of L. For every p_t there is a line (segment) intersecting p_t, which lies in the surface Σ, that is we have

$$\{r_t u + p_t | t \in U_1, u \in U_2\} \subset \Sigma,$$

where U_1 and U_2 are small subsets of $\mathbb{R}$. Continuity of the surface implies that r_t can be assumed to be continuous. Because if some sequence t_n converges to a t_l and r_{t_n} converges to $\tilde{r}_l \neq r_{t_l}$ we may replace r_{t_l} by $\tilde{r}_l$ due to the continuity of the surface.

From these considerations we may infer the following: Let Σ be a compact C^2 Riemannian surface having non-zero Gaussian curvature with boundary embedded isometrically in $\mathbb{R}^3$. If there is a set of line segments L_i which is dense in Σ and $|L_i| \geq l_{\min} > 0$ for every L_i, then Σ is a compact subset of a ruled surface. $l_{\min}$ will be referred to as the minimum length. This is clear due to the following; if the set L_i lies dense in M then we have that for every $x \in \Sigma$ there are two sequences namely a sequence of line segments L_n and points x_n on those segments which converge to x. Due to compactness we can find a convergent subsequence of line segments. We shall call its limit L and because L_i is a dense subset of Σ, the segment L must also lie on Σ and because the length of each line segment is greater than the minimum length L cannot shrink to a point, so x also lies on a line segment. Since the choice of x was arbitrary we find that for every point of Σ there is a line segment that lies on Σ and thus Σ is a compact subset of a ruled surface.

From our discussion above we can conclude that a surface with a dense set of lines is a ruled surface and may be parameterized as follows:

$$\sigma(u,t) = u\, r(t) + p(t),$$

where we can consider $r(t)$ and $p(t)$ curves on the surface. We can reparameterize σ so that $\|r(t)\| = 1$ and $r' \cdot p' = 0$. Clearly the first is achievable by redefining u. For the second part we follow Do Carmo [3] page 190 onwards; we wish to set $\langle p'(t), r'(t)\rangle = 0$. Note that we may add $v(t)r(t)$ to $p(t)$, where $v : \mathbb{R} \to \mathbb{R}$, leaving the surface in tact.[1] We write $\tilde{p}(t) = p(t) + v(t)r(t)$, and must choose $v(t)$ so that

$$\begin{aligned}\langle p'(t) + v'(t)r(t) + v(t)r'(t), r'(t)\rangle &= \langle p'(t), r'(t)\rangle + \langle v'(t)r(t), r'(t)\rangle + \langle v(t)r'(t), r'(t)\rangle \\ &= \langle p'(t), r'(t)\rangle + v(t)\langle r'(t), r'(t)\rangle \\ &= 0.\end{aligned}$$

Taking

$$v(t) = -\frac{\langle p'(t), r'(t)\rangle}{\langle r'(t), r'(t)\rangle}$$

clearly yields the correct result.

As we shall need the first and second fundamental forms of a ruled surface we remind ourselves of the following: For a general parameterized surface $\Sigma \subset \mathbb{R}^3$, parameterized by $\sigma : U \to \mathbb{R}^3$, where U is some subset of $\mathbb{R}^2$, we have:
The normal to Σ is given by

$$\nu = \frac{\partial_t \sigma \times \partial_u \sigma}{|\partial_t \sigma \times \partial_u \sigma|}.$$

[1] Note that if we consider a segment Σ of a ruled surface it may happen that after reparametrization the striction curve $p(t)$ no longer lies on the Σ but on the ruled surface of which Σ is a segment.

The second fundamental form is easily determined to be

$$
\begin{aligned}
\mathrm{II}(v,w) &= \langle \mathrm{d}\nu(v), w\rangle \\
&= \langle v(\nu), w\rangle \\
&= \left\langle v\left(\frac{\partial_t\sigma\times\partial_u\sigma}{|\partial_t\sigma\times\partial_u\sigma|}\right), w\right\rangle \\
&= \frac{1}{|\partial_t\sigma\times\partial_u\sigma|}\langle v(\partial_t\sigma\times\partial_u\sigma), w\rangle \\
&= \frac{1}{|\partial_t\sigma\times\partial_u\sigma|}\langle (v_u\partial_u\sigma^i\partial_i + v_t\partial_t\sigma^i\partial_i)(\epsilon^j_{kl}\partial_u\sigma^k\partial_t\sigma^l)\partial_j, w_u\partial_u\sigma^m\partial_m + w_t\partial_t\sigma^m\partial_m\rangle \\
&= \delta_{ij}\nu^j(w_uv_u\partial_u^2\sigma + (w_uv_t + w_tv_u)\partial_u\partial_t\sigma + w_tv_t\partial_t^2\sigma),
\end{aligned}
$$

where we expanded as follows $v_u\partial_u\sigma^i\partial_i + v_t\partial_t\sigma^i\partial_i = v$. This result can be rewritten as

$$
\mathrm{II} = \begin{pmatrix} \langle\nu,\partial_u^2\sigma\rangle & \langle\nu,\partial_u\partial_t\sigma\rangle \\ \langle\nu,\partial_u\partial_t\sigma\rangle & \langle\nu,\partial_t^2\sigma\rangle \end{pmatrix}.
$$

This result can also be found in [18] page 128.
Obviously the first fundamental form reads

$$
\mathrm{I} = \begin{pmatrix} \langle\partial_u\sigma,\partial_u\sigma\rangle & \langle\partial_u\sigma,\partial_t\sigma\rangle \\ \langle\partial_t\sigma,\partial_u\sigma\rangle & \langle\partial_t\sigma,\partial_t\sigma\rangle \end{pmatrix},
$$

using the same coordinates.
From this we may deduce that the Gaussian curvature is given by

$$
K = \frac{\det \mathrm{II}}{\det \mathrm{I}} = \frac{\det\begin{pmatrix} \langle\nu,\partial_u^2\sigma\rangle & \langle\nu,\partial_u\partial_t\sigma\rangle \\ \langle\nu,\partial_u\partial_t\sigma\rangle & \langle\nu,\partial_t^2\sigma\rangle \end{pmatrix}}{\det\begin{pmatrix} \langle\partial_u\sigma,\partial_u\sigma\rangle & \langle\partial_u\sigma,\partial_t\sigma\rangle \\ \langle\partial_t\sigma,\partial_u\sigma\rangle & \langle\partial_t\sigma,\partial_t\sigma\rangle \end{pmatrix}} = \frac{\langle\nu,\partial_u^2\sigma\rangle\langle\nu,\partial_t^2\sigma\rangle - \langle\nu,\partial_u\partial_t\sigma\rangle^2}{\langle\partial_u\sigma,\partial_u\sigma\rangle\langle\partial_t\sigma,\partial_t\sigma\rangle - \langle\partial_u\sigma,\partial_t\sigma\rangle^2}.
$$

Applying these formulae to a ruled surface parameterized as discussed above yields

$$
\nu = \frac{(ur'+p')\times r}{|(ur'+p')\times r|} = \frac{(ur'+p')\times r}{\sqrt{u^2r^2(r')^2 + (p'\times r)^2}} = \frac{(ur'+p')\times r}{\sqrt{u^2(r')^2 + (p'\times r)^2}}.
$$

Since $\partial_u^2\sigma(u,t) = 0$ the Gaussian curvature simplifies to

$$
K = -\frac{\langle\nu,\partial_u\partial_t\sigma\rangle^2}{|(ur'+p')\times r|^2} = -\left(\frac{r'\cdot(p'\times r)}{|(ur'+p')\times r|}\right)^2\frac{1}{|(ur'+p')\times r|^2} = -\frac{(r')^2(p'\times r)^2}{(u^2(r')^2 + (p'\times r)^2)^2}. \quad (2)
$$

We can also easily deduce that the volume element is given by

$$
\sqrt{u^2|r|^2|r'|^2 + |r\times p'|^2}\mathrm{d}u\mathrm{d}t
$$

We now focus on the following problem; how does the normal alter along a ruling. The result we derive may is a corollary of lemma 2 but may also be seen as an extension of the following lemma (lemma 5.6.6 of [15]):

Lemma 1 *A ruled surface Σ has Gaussian curvature $K \leq 0$. Furthermore, $K = 0$ if and only if the unit normal ν is parallel along each ruling of Σ.*

Lemma 2 *Let γ be an asymptotic curve, T its unit tangent, and U the normal to the surface. Then the shape operator satisfies $S(T) = \sqrt{|K|}U \times T$.*

Proof Consider the Darboux frame T, V, U of γ where $V = U \times T$. We have the Darboux equations

$$\begin{aligned} T' &= \qquad gV + kU \\ V' &= -gT \qquad + tU \\ U' &= -kT - tV, \end{aligned}$$

where $k = S(T) \cdot T$ is the normal curvature $k(T)$ in the T direction. Because γ is assumed to be asymptotic we have that $k = 0$. This implies that if we use the basis T, V the shape operator is of the form

$$S = \begin{pmatrix} 0 & * \\ t & * \end{pmatrix}.$$

Symmetry of the shape operator yields

$$S = \begin{pmatrix} 0 & t \\ t & * \end{pmatrix}.$$

This in turn yields that $K = -|K| = \det(S) = -t^2$. So that $S(T) = \sqrt{|K|}V = \sqrt{|K|}U \times T$. □

Corollary 3 *Let Σ be a ruled surface, parameterized as above. The derivative of the normal along the ruling $\partial_u \nu$, usually identified with the image of r under the shape shape operator of Σ at $p(t)$ $S(r)$, satisfies $S(r) = \sqrt{|K|}n$, where $n = \nu \times r$*

Proof Apply lemma 2 to Σ, with $\gamma(u) = ur(t_0)$ for some fixed t_0. □

Hausdorff distance

In the introduction be already mentioned that the Hausdorff distance will be the measure of accuracy of an approximation. In the following subsection we give some definitions and elementary results.

The Hausdorff distance between two subsets of a metric space can be defined as follows

$$d^H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y|\},$$

this definition is the most common in the field of approximations of surfaces [2, 8, 16]. Equivalently we can also define

$$d^H(A,B) = \inf\{\epsilon | A \subset U(B,\epsilon) \text{ and } B \subset U(A,\epsilon)\},$$

where $U(A,\epsilon)$ denotes the ϵ-neighbourhood of A, the latter form can be found in Munkres [14] page 280. It is not difficult to see that both definitions are equivalent, because if ϵ is such that $A \subset U(B,\epsilon)$ and $B \subset U(A,\epsilon)$, the we have that for all $x \in A$ there exists a $y \in B$ such that $|x-y| < \epsilon$ as well as $y \in B$ there exists a $x \in A$ such that $|x-y| < \epsilon$. This in turn is equivalent to

$$\sup_{x\in A}\inf_{y\in B}|x-y| \leq \epsilon \text{ and } \sup_{y\in B}\inf_{x\in A}|x-y| \leq \epsilon.$$

So that

$$d^H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\} = \inf\{\epsilon | A \subset U(B,\epsilon) \text{ and } B \subset U(A,\epsilon)\}.$$

The one-sided Hausdorff distance is defined to be

$$d_o^H(A,B) = \sup_{x\in A}\inf_{y\in B}|x-y| = \inf\{\epsilon | A \subset U(B,\epsilon)\}.$$

The one-sided Hausdorff distance satisfies the following condition

$$d_o^H(A,C) \leq d_o^H(A,B) + d_o^H(B,C).$$

Because if we have that $A \subset U(B,\epsilon)$ and $B \subset U(C,\tilde{\epsilon})$ then $A \subset U(C,\epsilon+\tilde{\epsilon})$ and thus

$$\inf\{\epsilon | A \subset U(C,\epsilon)\} \leq \inf\{\epsilon | A \subset U(B,\epsilon)\} + \inf\{\epsilon | B \subset U(C,\epsilon)\}.$$

This argument also yields the triangle inequality for the Hausdorff distance.

Having given these definitions we may prove the following lemma:

Lemma 4 *Given a surface Σ embedded in $\mathbb{R}^m$. Now consider one-dimensional simplices (edges) whose vertices lie on the surface Σ. If we identify a one dimensional simplex (edge) E in $\mathbb{R}^m$ with its vertices v_1 and v_2 then the function*

$$d_o^H(\cdot,\Sigma) : \mathbb{R}^{2m} \to \mathbb{R} : (v_1,v_2) \mapsto d_o^H(v_1v_2,\Sigma)$$

induced by the Hausdorff distance is continuous.

Proof If $|(v_1,v_2) - (\tilde{v}_1,\tilde{v}_2)| \leq \delta$ then clearly we have $d_o^H(v_1v_2,\tilde{v}_1\tilde{v}_2) \leq 2\delta$ and $d_o^H(\tilde{v}_1\tilde{v}_2,v_1v_2) \leq 2\delta$. This gives us $|d_o^H(v_1v_2,\Sigma) - d_o^H(\tilde{v}_1\tilde{v}_2,\Sigma)| \leq 4\delta$. So $d_o^H(\cdot,\Sigma)$ is continuous. □

This result can be put to use as follows:

Lemma 5 *Suppose Σ is a compact connected smooth subset of a smooth ruled surface, then we have that if a family of one dimensional simplices E_n satisfies*

$$\lim_{n\to\infty} d_o^H(E_n, \Sigma) = 0,$$

and the simplices E_n converge to one E, then there is a ruling R of Σ such that

$$\lim_{n\to\infty} d_o^H(E_n, R) = 0.$$

Proof We first note that $d_o^H(E, \Sigma) = 0$ implies that $E \subset \Sigma$ and thus E is a segment of a ruling, say R. Continuity, in the sense of Cauchy, now implies that if

$$\lim_{n\to\infty} d_o^H(E_n, \Sigma) = 0,$$

then

$$\lim_{n\to\infty} d_o^H(E_n, R) = 0.$$

□

Let us now we assume that the compact ruled surface is not a subset of the plane. We can then conclude that if we have a sequence of triangles t^i, with vertices v_1^i, v_2^i, v_3^i, for which

$$\lim_{i\to\infty} d_o^H(t_i, \Sigma) = 0,$$

then for every t^i there exists a ruling R^i such that

$$\lim_{i\to\infty} d_o^H(t^i, R^i) = 0$$

This can be seen as follows; any sequence on a closed and bounded set has a convergent subsequence. So the sequence of three tuples (v_1^i, v_2^i, v_3^i) has a convergent subsequence. Consider any convergent subsequence. We have for every edge $v_a^i v_b^i$ convergence to some $v_a v_b$ and $d_o^H(v_a^i v_b^i, \Sigma)$ tends to zero, because we assume that the one-sided Hausdorff distance of the entire triangle tends to zero. This implies that $d_o^H(v_a^i v_b^i, R)$ tends to zero for one particular ruling. Because our argument does not depend on a, b we have that the entire triangle must lie close to the ruling, since we assumed that the ruled surface is not the plane.

This discussion inspires the following definitions:

Definition 6 *A sequence of triangles with*

$$\lim_{i\to\infty} d_o^H(t^i, \Sigma) = 0$$

is called asymptotically line-like if the length of at least one of the edges of t^i does not tend to zero. It is called asymptotically point-like if the length of all edges tends to zero.

Asymptotically line-like triangulations

Fejes Tóth states in section 12 of chapter 5 of [6] the following: 'Let T be the one-sheeted hyperboloid bounded by two congruent circles A and B. We inscribe A and B by regular n-polygons $A_1 \dots A_n$ and $B_1 \dots B_n$ respectively, so that $A_1B_1, \dots, A_nB_n$ lie on the hyperboloid. The polyhedron-like surface F_{2n} is best described by its faces $A_1A_2B_1, \dots, A_nA_1B_n$ and $B_1B_2A_2, \dots, B_nB_1A_1$. The deviation $d^H(F_{2n}, T)$ is determined by the deviation of the n-polygon $A_1 \dots A_n$ from the circle A, this implies that order of magnitude of the deviation is $1/n^2$ and not $1/n$.'
We shall see that this statement is not correct. As a consequence of lemma 3; the normal to a ruled surface, in this case the hyperboloid, varies along a ruling and thus the maximum is not attained near the two congruent circles, but halfway across the ruling.[2] In fact it can be verified that the triangulation suggested by Fejes Tóth is poorer then a triangulation[3] such as Clarkson [2] constructs in his work on triangulation of non-convex surfaces. This will be discussed in greater detail below.

The study of the hyperboloid leads to the following conjecture:

Conjecture 7 *Let Σ be a compact manifold of nonzero Gaussian curvature, possibly with a boundary and let T_n be the sequence of optimal triangulations with n vertices, where each T_n is assumed to be isotopic to Σ. Then (almost) every sequence t^n, where $t^n \in T_n$ is asymptotically point-like.*

The proof of this conjecture will generate the right notion of 'almost' in this context. For now, we think of the following condition

$$U_n = \bigcup_{t_n \in \mathrm{AL}} t_n,$$

where AL denotes the set of asymptotically line-like triangles. In other words, the conjecture states that the limit of the union of all asymptotically line-like triangles has measure zero.

In connection with conjecture 7 we note the following corollary of lemma 4, which also may be seen as an argument in favour of the conjecture.

Corollary 8 *Let Σ be a compact manifold, which contains no straight lines and let T_n be the sequence of optimal triangulations with n vertices, where each T_n is assumed to be isotopic to Σ. Then every sequence t^n, where $t^n \in T_n$ is asymptotically point-like.*

Proof We shall follow the same argument as in the proof of lemma 5. Suppose there is a sequence t^n which is not asymptotically point-like. Then consider the edges E_i^n, $i = 1, 2, 3$, of t^n and choose a

[2]This is a consequence of the reflection symmetry exhibited by the one-sheeted hyperboloid bounded by two congruent circles.
[3]These triangulations are asymptotically point-like.

convergent subsequence $E^n_{i(n)}$ whose limit we shall denote by E and whose length is strictly greater then zero. Because t^n is assumed asymptotically line-like this is possible. Due to the fact that t^n is an optimal triangulation we have that

$$\lim_{n\to\infty} d_o^H(E_n^{i(n)}, \Sigma) = 0$$

and thus $E \subset \Sigma$. So Σ contains a straight line, a clear contradiction with the hypothesis. □

To prove this conjecture it would be sufficient to show that for a sequence of triangulations isomorphic to the surface that for each individual triangle t in a sequence of asymptotically line-like triangles we have that

$$\frac{1}{\sqrt{27}} \int \sqrt{|K|} \mathrm{d}\sigma_t \leq \frac{1}{2} d_o^H(\Sigma, t),$$

where σ_t is the image of the projection along the normal of the surface of t. Because if we now sum over all triangles we find, as n tends to infinity, that

$$\begin{aligned}\frac{1}{\sqrt{27}} \int \sqrt{|K|} \mathrm{d}\Sigma &\leq \frac{1}{\sqrt{27}} \sum_{t\in T} \int \sqrt{|K|} \mathrm{d}\sigma_t \\ &\leq \sum_{t\in T} \frac{1}{2} d_o^H(\Sigma, t) \\ &\leq n \sup_{t\in T} d_o^H(\Sigma, t) \\ &\leq n d^H(T, \Sigma), \end{aligned} \tag{3}$$

where we used in the first line that T is isometric to Σ, the second line comprises the conjecture, the third line relies on the fact that the number of triangles behaves like $2n$ as n tends to infinity, while the final line follows from the definition of the Hausdorff distance. Equation (3) yields the conjecture because Clarkson [2] sketches a construction of a triangulation for which

$$n d^H(T, \Sigma) \leq \frac{1}{\sqrt{27}} \int \sqrt{|K|} \mathrm{d}\Sigma.$$

Furthermore each sequence of triangles in the triangulation proposed by Clarkson is asymptotically point-like. This implies that a sequence of triangulations of which the asymptotically line-like triangles lie densely on the surface cannot be asymptotically optimal.

We shall now focus on asymptotically line-like triangle sequences on a ruled surface. We can use that these triangles lie close to a ruling to approximate the ruled surface. In particular we shall use that we can use a Taylor series to describe $r(t)$ and $p(t)$. We shall work in the neighbourhood of $t = 0$ and so that the normal to the surface is

$$\nu(u, t) = \frac{(ur'(0) + p'(0)) \times r(0)}{\sqrt{u^2(r'(0))^2 + (p'(0) \times r(0))^2}} + \mathcal{O}(t),$$

which means that the surface can be reparameterized by

$$\sigma(u,\tau) = ur(0) + \tau(r(0)\times\nu(u,0)) + \mathcal{O}(\tau^2). \tag{4}$$

Here we slightly abuse notation in the following manner; actually we have that

$$\sigma(u,t) = p(t) + ur(t) \simeq p'(0)t + ur(0) + ur'(0)t,$$

where we assume that $p(0) = 0$. On the other hand

$$\begin{aligned}
\sigma(\tilde{u},\tau) &= \tilde{u}r + \tau(r\times\nu(\tilde{u},0)) + \mathcal{O}(\tau^2) \\
&= \tilde{u}r + \tau(r\times\frac{(\tilde{u}r'+p')\times r}{|\nu(\tilde{u},0)|})) + \mathcal{O}(\tau^2) \\
&= \left(\tilde{u} - \frac{1}{|\nu(\tilde{u},0)|}r\cdot(\tilde{u}r'+p')\right)r + \frac{r^2}{|\nu(\tilde{u},0)|}\tau(\tilde{u}r'+p') + \mathcal{O}(\tau^2)
\end{aligned}$$

So that

$$\begin{aligned}
u &= \tilde{u} - \frac{r\cdot(\tilde{u}r'+p')}{|\nu(\tilde{u},0)|}\tau + \mathcal{O}(\tau^2) \\
t &= \frac{r^2}{|\nu(\tilde{u},0)|}\tau + \mathcal{O}(\tau^2).
\end{aligned}$$

It obviously does not matter if there is a correction of u of order t, so this justifies our abuse of notation.

We can now pick three points which will act as vertices $v_1 \simeq u_1r(0) + \tau_1(r(0)\times\nu(u_1,0))$, $v_2 \simeq u_2r(0)+\tau_2(r(0)\times\nu(u_2,0))$ and $v_3 \simeq u_3r(0)+\tau_3(r(0)\times\nu(u_3,0))$. Here and from this point onwards $\simeq$ means equality up to first order in τ. Naturally the triangle with these vertices can be parameterized by

$$\{(v_1-v_3)\lambda_1 + (v_2-v_3)\lambda_2 + v_3 | \lambda_1\in[0,1], \lambda_2\in[0,1-\lambda_1]\}.$$

To be able to calculate the one-sided Hausdorff distance of the triangle to the surface we need to calculate the distance of a given point q of the triangle to the surface. That is we need to find ρ such that

$$\rho\nu(u,0) + ur(0) + \tau(r(0)\times\nu(u,0)) \simeq q,$$

where we note that $\{\nu(u,0), r(0), r(0)\times\nu(u,0)\}$ is an orthonormal system. Furthermore we have

$$\begin{aligned}
q \simeq&(u_1r(0) + \tau_1(r(0)\times\nu(u_1,0)) - u_3r(0) - \tau_3(r(0)\times\nu(u_3,0)))\lambda_1 \\
&+ (u_2r(0) + \tau_2(r(0)\times\nu(u_2,0)) - u_3r(0) - \tau_3(r(0)\times\nu(u_3,0)))\lambda_2 \\
&+ u_3r(0) + \tau_3(r(0)\times\nu(u_3,0))
\end{aligned}$$

From $q \cdot r(0) = u$ we find that $u = (u_1 - u_3)\lambda_1 + (u_2 - u_3)\lambda_2 + u_3$, likewise we find

$$
\begin{aligned}
|\rho| = |q \cdot \nu(u,0)| \simeq &|((\tau_1(r(0) \times \nu(u_1,0)) - \tau_3(r(0) \times \nu(u_3,0)))\lambda_1 \\
&+ (\tau_2(r(0) \times \nu(u_2,0)) - \tau_3(r(0) \times \nu(u_3,0)))\lambda_2 \\
&+ \tau_3(r(0) \times \nu(u_3,0))) \cdot \nu(u,0)|
\end{aligned}
$$

We first calculate

$$
\begin{aligned}
[r(0) \times \nu(u_1,0)] \cdot \nu(u,0) &= \left(r(0) \times \frac{(u_1 r'(0) + p'(0)) \times r(0)}{\sqrt{u_1^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \right) \frac{(u r'(0) + p'(0)) \times r(0)}{\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \\
&= \frac{(u_1 r'(0) + p'(0)) \cdot [(u r'(0) + p'(0)) \times r(0)]}{\sqrt{u_1^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \\
&= \frac{\det((u_1 r'(0) + p'(0)) \quad (u r'(0) + p'(0)) \quad r(0))}{\sqrt{u_1^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \\
&= \frac{(u_1 - u)\det(r'(0) \quad p'(0) \quad r(0))}{\sqrt{u_1^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \\
&= \frac{(u_1 - u)(p'(0) \times r(0)) \cdot r'(0)}{\sqrt{u_1^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}},
\end{aligned}
$$

where we once more used that $(a \times b) \times c = b(a \cdot c) - a(b \cdot c)$ in the first line. So that

$$
\begin{aligned}
|\rho| \simeq &\Bigg| \frac{(u_1 - u)\tau_1\lambda_1}{\sqrt{u_1^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \\
&- \frac{(u_3 - u)\tau_3\lambda_1}{\sqrt{u_3^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \\
&+ \frac{(u_2 - u)\tau_2\lambda_2}{\sqrt{u_2^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \\
&- \frac{(u_3 - u)\tau_3\lambda_2}{\sqrt{u_3^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \\
&+ \frac{(u_3 - u)\tau_3}{\sqrt{u_3^2 (r'(0))^2 + (p'(0) \times r(0))^2}\sqrt{u^2 (r'(0))^2 + (p'(0) \times r(0))^2}} \Bigg| |p'(0) \times r(0)| \cdot |r'(0)| \\
= &\sqrt[4]{|K|(u,0)} \Big| \sqrt[4]{|K|(u_1,0)}(u_1 - u)\lambda_1\tau_1 + \sqrt[4]{|K|(u_2,0)}(u_2 - u)\lambda_2\tau_2 \\
&+ \sqrt[4]{|K|(u_3,0)}(u_3 - u)(1 - \lambda_1 - \lambda_2)\tau_3 \Big|,
\end{aligned}
$$

where we used (2). We therefore conclude that

$$
\begin{aligned}
D_o^H(\triangle v_1 v_2 v_3, \Sigma) \simeq \sup_{(\lambda_1,\lambda_2) \in \Lambda} \sqrt[4]{|K|(u,0)} \Big| &\sqrt[4]{|K|(u_1,0)}(u_1 - u)\lambda_1\tau_1 + \sqrt[4]{|K|(u_2,0)}(u_2 - u)\lambda_2\tau_2 \\
&+ \sqrt[4]{|K|(u_3,0)}(u_3 - u)(1 - \lambda_1 - \lambda_2)\tau_3 \Big|,
\end{aligned}
$$

where $u = (u_1 - u_3)\lambda_1 + (u_2 - u_3)\lambda_2 + u_3$ and $\Lambda = \{(\lambda_1, \lambda_2) | \lambda_1 \in [0,1], \lambda_2 \in [0, 1-\lambda_1]\}$. We now introduce the notation $\delta_i = \sqrt[4]{|K|(u_i, 0)}\tau_i$ so that we may write

$$D_o^H(\triangle v_1 v_2 v_3, \Sigma) \simeq \sup_{(\lambda_1,\lambda_2)\in\Lambda} \sqrt[4]{|K|(u,0)} \Big|(u_1 - u)\lambda_1\delta_1 + (u_2 - u)\lambda_2\delta_2 + (u_3 - u)(1 - \lambda_1 - \lambda_2)\delta_3\Big|,$$

It can be verified using Mathematica that the only critical point of

$$\sqrt[4]{|K|(u,0)}\Big((u_1 - u)\lambda_1\delta_1 + (u_2 - u)\lambda_2\delta_2 + (u_3 - u)(1 - \lambda_1 - \lambda_2)\delta_3\Big)$$

is a saddle point and thus the supremum is attained on the boundary of the triangle. The supremum on each of the edges of the elementary triangle is easily determined because on the vertices the value of the function is zero, the part of the expression of which we take the absolute value is ether positive or negative semi-definite, and it turns out that there is but one critical point on the edge. More explicitly, if we assume that $\lambda_2 = 0$, we naturally have that

$$\begin{aligned}\sup_{\lambda_1\in[0,1],\lambda_2=0} &\sqrt[4]{|K|(u,0)}\Big|\delta_1(u_1 - u)\lambda_1 + \delta_2(u_2 - u)\lambda_2 + \delta_3(u_3 - u)(1 - \lambda_1 - \lambda_2)\Big| \\ &= \sup_{\lambda_1\in[0,1]} \sqrt[4]{|K(u,0)|}|(u_1 - u)\delta_1\lambda_1 + (u_3 - u)\delta_3\lambda_3|\end{aligned}$$

with $u = (u_1 - u_3)\lambda_1 + u_3$ and $\lambda_3 = 1 - \lambda_1$, can be rewritten in the form

$$\begin{aligned}\sup_{u\in[u_1,u_3]} &\sqrt[4]{|K(u,0)|}\Big|\frac{u - u_3}{u_1 - u_3}(u_1 - u)\delta_1 + \frac{u_1 - u}{u_1 - u_3}\delta_3(u_3 - u)\Big| \\ &= \sup_{u\in[u_1,u_3]} \sqrt[4]{|K(u,0)|}\Big|\frac{\delta_1 - \delta_3}{u_1 - u_3}\Big||(u - u_3)(u_1 - u)| \\ &= \sup_{u\in[u_1,u_3]} \frac{1}{\sqrt{cu^2 + c^{-1}}}\Big|\frac{\delta_1 - \delta_3}{u_1 - u_3}\Big||(u - u_3)(u_1 - u)| \\ &= \frac{1}{\sqrt{cu_c^2 + c^{-1}}}\Big|\frac{\delta_1 - \delta_3}{u_1 - u_3}\Big||(u_c - u_3)(u_1 - u_c)|,\end{aligned}$$

where we used equation (2) and have that

$$\begin{aligned}u_c =& \frac{2^{1/3}\left(-2 + c^2u_1u_3\right)}{\left(27c^4(u_1 + u_3) + \sqrt{729c^8(u_1 + u_3)^2 + 4\left(6c^2 - 3c^4u_1u_3\right)^3}\right)^{1/3}} \\ &+ \frac{\left(27c^4(u_1 + u_3) + \sqrt{729c^8(u_1 + u_3)^2 + 4\left(6c^2 - 3c^4u_1u_3\right)^3}\right)^{1/3}}{3\cdot 2^{1/3}c^2} \\ =& \frac{2^{1/3}\left(-2 + c^2u_1u_3\right)}{\left(27c^4(u_1 + u_3) + \sqrt{-27c^4\Delta}\right)^{1/3}} \\ &+ \frac{\left(27c^4(u_1 + u_3) + \sqrt{-27c^4\Delta}\right)^{1/3}}{3\cdot 2^{1/3}c^2},\end{aligned}$$

the solution to the equation

$$c^2u^3 + (2 - c^2u_1u_3)u - (u_1 + u_3) = 0,$$

whose discriminant is

$$\Delta = -27c^4(u_1 + u_3)^2 - 4c^2(2 - c^2u_1u_3)^3$$

and

$$c = \frac{|r'|}{|p' \times r|}.$$

It turns out that there are some minor problems with the definition of the root as given above. Because we take a third order root. There is a jump between real roots at

$$27c^4(u_1 + u_3) + \sqrt{729c^8(u_1 + u_3)^2 + 4\left(6c^2 - 3c^4u_1u_3\right)^3} = 0,$$

due to the branch cut in the standard definition of the third power root, that is if

$$u_1 = \frac{2}{c^2u_3}, \tag{5}$$

and $u_3 < 0$. It might be best to define the (relevant) real solution as above if $u_1 \geq -u_3$ and $u_3 \geq u_1$ and elsewhere via the invariance of u under the interchanging of u_1 and u_3 and switching of sign of u if u_1 and u_3 do so. As a footnote we remark that $c^2u^3 + (2 - c^2u_1u_3)u - (u_1 + u_3) = 0$ if a submersion at every point of the surface in $\mathbb{R}^3$ and therefore a smooth surface consisting of three connected components.

This completes our discussion of the Hausdorff distance for now and we focus on the determination of $\mathrm{d}\sigma_t$. We again use the parameterizations

$$\begin{aligned}
q(\lambda_1, \lambda_2) &= (v_1 - v_3)\lambda_1 + (v_2 - v_3)\lambda_2 + v_3 \\
v_i &= u_i r(0) + \tau_i(r(0) \times \nu(u, 0)) \\
\nu(u, 0) &= \frac{(ur' + p') \times r}{\sqrt{u^2r'^2 + (p' \times r)^2}} \\
\sigma(u, \tau) &= ur(0) + \tau(r(0) \times \nu(u, 0)) + \mathcal{O}(\tau^2).
\end{aligned}$$

As we did for the determination of ρ, we use projections to find the projection of the triangle over the normal of the surface denoted by σ_t

$$\sigma_t(\lambda_1, \lambda_2) \simeq (q(\lambda_1, \lambda_2) \cdot r)r + (q(\lambda_1, \lambda_2) \cdot (r(0) \times \nu(u(\lambda_1, \lambda_2), 0)))(r(0) \times \nu(u(\lambda_1, \lambda_2), 0)).$$

Clearly

$$u = q \cdot r = (u_1 - u_3)\lambda_1 + (u_2 - u_3)\lambda_2 + u_3.$$

To the determine the other innerproduct we first calculate the following

$$\begin{aligned}
v_i\cdot(r\times\nu(u,0)) &= (u_ir+\tau_i(r\times\nu(u_i,0)))\cdot\frac{r\times((r'u+p')\times r)}{\sqrt{u^2r'^2+(p'\times r)^2}}\\
&=\tau_i\frac{(r\times\nu(u_i,0))\cdot((r'u+p')r^2-r(p'\cdot r'))}{\sqrt{u^2r'^2+(p'\times r')^2}}\\
&=\tau_i\frac{(r\times(r'u_i+p'))\cdot((r'u+p')r^2)}{\sqrt{u_ir'^2+(p'\times r)^2}\sqrt{ur'^2+(p'\times r)^2}}\\
&=\tau_i\frac{(r\times(r'u_i+p'))\cdot(r+p')}{\sqrt{u_i^2r'^2+(p'\times r)^2}\sqrt{u^2r'^2+(p'\times r)^2}}\\
&=\tau_i\frac{(u_i-u)r\cdot(r'\times p')}{\sqrt{u_i^2r'^2+(p'\times r)^2}\sqrt{u^2r'^2+(p'\times r)^2}}\\
&=\tau_i\frac{(u_i-u)r\cdot(r'\times p')}{\sqrt{u_i^2r'^2+(p'\times r)^2}\sqrt{u^2r'^2+(p'\times r)^2}}\\
&=\frac{r'\cdot(r\times p')}{|r'||r\times p'|}\tau_i(u_i-u)\sqrt[4]{|K(u_i,0)|}\sqrt[4]{|K(u,0)|},
\end{aligned}$$

where u is as above. So that

$$\begin{aligned}
q\cdot(r\times\nu(u,0)) &= ((v_1-v_3)\lambda_1+(v_2-v_3)\lambda_2+v_3)\cdot(r\times\nu(u,0))\\
&=\frac{r'\cdot(r\times p')}{|r'||r\times p'|}\sqrt[4]{|K(u,0)|}(\tau_1(u_1-u)\sqrt[4]{|K(u_1,0)|}\lambda_1+\tau_2(u_2-u)\sqrt[4]{|K(u_2,0)|}\lambda_2\\
&\qquad+\tau_3(u_3-u)\sqrt[4]{|K(u_3,0)|}(1-\lambda_1-\lambda_2)).
\end{aligned}$$

Which in turn gives

$$\begin{aligned}
\sigma_t(\lambda_1,\lambda_2)\simeq ur+\frac{r'\cdot(r\times p')}{|r'||r\times p'|}\sqrt[4]{|K(u,0)|}(\tau_1(u_1-u)\sqrt[4]{|K(u_1,0)|}\lambda_1\\
+\tau_2(u_2-u)\sqrt[4]{|K(u_2,0)|}\lambda_2\\
+\tau_3(u_3-u)\sqrt[4]{|K(u_3,0)|}(1-\lambda_1-\lambda_2))(r\times\nu(u,0)).
\end{aligned}$$

This defines a surface segment embedded in $\mathbb{R}^3$ of which we need to calculate the surface, for example by using the standard integral

$$\text{area}(\{\sigma_t(\lambda_1,\lambda_2)|\lambda_1\in[0,1],\lambda_2\in[0,1-\lambda_1]\})=\int_0^1\int_0^{1-\lambda_1}|\partial_{\lambda_1}\sigma_t(\lambda_1,\lambda_2)\times\partial_{\lambda_2}\sigma_t(\lambda_1,\lambda_2)|\mathrm{d}\lambda_1\mathrm{d}\lambda_2.$$

The calculation may be simplified considerably by the following observation, we may write

$$\begin{aligned}
\partial_{\lambda_i}\sigma_t(\lambda_1,\lambda_2) &= (\partial_{\lambda_i}u)r+\partial_u(\sqrt[4]{|K|})g(\lambda_1,\lambda_2)(\partial_{\lambda_i}u)r\times\nu\\
&\quad+\sqrt[4]{|K|}\partial_{\lambda_i}g(\lambda_1,\lambda_2)r\times\nu\\
&\quad+\sqrt[4]{|K|}g(\lambda_1,\lambda_2)(\partial_{\lambda_i}u)r\times\partial_u\nu
\end{aligned}$$

we can now see that the terms involving $\partial_{\lambda_i} u$ vanish because $\partial_{\lambda_1} u \partial_{\lambda_2} u$ is symmetric and such terms cancel out in an exterior product.
We first focus on

$$\begin{aligned}\partial_{\lambda_1}\sigma_t(\lambda_1,\lambda_2) \simeq \partial_{\lambda_1}(u)r + \frac{r'\cdot(r\times p')}{|r'||r\times p'|}\sqrt[4]{|K(u,0)|}\partial_{\lambda_1}(&\tau_1(u_1-u)\sqrt[4]{|K(u_1,0)|}\lambda_1 \\ &+\tau_2(u_2-u)\sqrt[4]{|K(u_2,0)|}\lambda_2 \\ &+\tau_3(u_3-u)\sqrt[4]{|K(u_3,0)|}(1-\lambda_1-\lambda_2))(r\times\nu(u,0)) \\ &+\text{symmetric terms}\end{aligned}$$

So that

$$\begin{aligned}&\partial_{\lambda_1}\sigma_t(\lambda_1,\lambda_2) \simeq (u_1-u_3)r \\ &\quad+\frac{r'\cdot(r\times p')}{|r'||r\times p'|}\sqrt[4]{|K(u,0)|}(\tau_1\sqrt[4]{|K(u_1,0)|}((u_1-u)-(u_1-u_3)\lambda_1) \\ &\qquad\qquad -\tau_2(u_1-u_3)\sqrt[4]{|K(u_2,0)|}\lambda_2 \\ &\qquad\qquad -\tau_3\sqrt[4]{|K(u_3,0)|}((u_1-u_3)(1-\lambda_1-\lambda_2)+(u_3-u)))(r\times\nu(u,0)) \\ &\qquad\qquad +\text{symmetric terms.}\end{aligned}$$

Likewise we find that

$$\begin{aligned}&\partial_{\lambda_2}\sigma_t(\lambda_1,\lambda_2) \simeq (u_2-u_3)r \\ &\quad+\frac{r'\cdot(r\times p')}{|r'||r\times p'|}\sqrt[4]{|K(u,0)|}(-\tau_1(u_2-u_3)\sqrt[4]{|K(u_1,0)|}\lambda_1 \\ &\qquad\qquad +\tau_2\sqrt[4]{|K(u_2,0)|}((u_2-u)-(u_2-u_3)\lambda_2) \\ &\qquad\qquad -\tau_3\sqrt[4]{|K(u_3,0)|}((u_2-u_3)(1-\lambda_1-\lambda_2)+(u_3-u)))(r\times\nu(u,0)) \\ &\qquad\qquad +\text{symmetric terms.}\end{aligned}$$

Using that r, ν and $r \times \nu$ form an orthogonal basis of $\mathbb{E}^3$ we find

$$\begin{aligned}
|\partial_{\lambda_1}\sigma_t \times \partial_{\lambda_2}\sigma_t| = |((u_1-u_3)\sqrt[4]{|K(u,0)|}(&-\tau_1(u_2-u_3)\sqrt[4]{|K(u_1,0)|}\lambda_1\\
&+\tau_2\sqrt[4]{|K(u_2,0)|}((u_2-u)-(u_2-u_3)\lambda_2)\\
&-\tau_3\sqrt[4]{|K(u_3,0)|}((u_2-u_3)(1-\lambda_1-\lambda_2)+(u_3-u))))\\
-((u_2-u_3)\sqrt[4]{|K(u,0)|}(&\tau_1\sqrt[4]{|K(u_1,0)|}((u_1-u)-(u_1-u_3)\lambda_1)\\
&-\tau_2(u_1-u_3)\sqrt[4]{|K(u_2,0)|}\lambda_2\\
&-\tau_3\sqrt[4]{|K(u_3,0)|}((u_1-u_3)(1-\lambda_1-\lambda_2)+(u_3-u))))|\\
= \sqrt[4]{|K(u,0)|}\,|((u_1-u_3)(&\tau_2\sqrt[4]{|K(u_2,0)|}(u_2-u)\\
&-\tau_3\sqrt[4]{|K(u_3,0)|}(u_3-u)))\\
-((u_2-u_3)(&\tau_1\sqrt[4]{|K(u_1,0)|}(u_1-u)\\
&-\tau_3\sqrt[4]{|K(u_3,0)|}(u_3-u)))|\\
= \sqrt[4]{|K(u,0)|}\,|&(u_1-u_3)\tau_2\sqrt[4]{|K(u_2,0)|}(u_2-u)\\
&-(u_1-u_2)\tau_3\sqrt[4]{|K(u_3,0)|}(u_3-u)\\
&-(u_2-u_3)\tau_1\sqrt[4]{|K(u_1,0)|}(u_1-u)|\\
= \sqrt[4]{|K(u,0)|}\,|&(u_1-u_3)\delta_2(u_2-u)-(u_1-u_2)\delta_3(u_3-u)-(u_2-u_3)\delta_1(u_1-u)|\\
= \sqrt[4]{|K(u,0)|}\,|&(u_1-u_2)(\delta_2-\delta_3)(u_3-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|
\end{aligned}$$

This gives us that

$$\begin{aligned}
&\operatorname{area}(\{\sigma_t(\lambda_1,\lambda_2)|\lambda_1\in[0,1],\lambda_2\in[0,1-\lambda_1]\})\\
&\quad=\int_0^1\int_0^{1-\lambda_2}\sqrt[4]{|K(u,0)|}\,|(u_1-u_3)\delta_2(u_2-u)-(u_1-u_2)\delta_3(u_3-u)-(u_2-u_3)\delta_1(u_1-u)|\mathrm{d}\lambda_1\mathrm{d}\lambda_2\\
&\quad=\int_0^1\int_0^{1-\lambda_2}\sqrt[4]{|K(u,0)|}\,|(u_1-u_2)(\delta_2-\delta_3)(u_3-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|\mathrm{d}\lambda_1\mathrm{d}\lambda_2.
\end{aligned} \tag{6}$$

We now describe a way of dealing with the integral (6). To do so we shall first assume that $u_1 \le u_2 \le u_3$. Apart from $u = (u_1-u_3)\lambda_1+(u_2-u_3)\lambda_2+u_3$ we introduce $v = (u_1-u_3)\lambda_1-(u_2-u_3)\lambda_2$ as our new coordinates instead of λ_1 and λ_2, because u is the natural coordinate for $|K(u,0)|$. Clearly the determinant of the Jacobian is in this case given by

$$\begin{vmatrix} u_1-u_3 & u_2-u_3 \\ u_1-u_3 & -u_2+u_3 \end{vmatrix} = -2(u_1-u_3)(u_2-u_3).$$

Furthermore thanks to the assumption $u_1 \le u_2 \le u_3$ it is easy to see that $u_1 \le u \le u_3$ on the other hand the inequalities

$$\begin{aligned}
0 \le \lambda_1 \le 1\\
0 \le \lambda_2 \le 1-\lambda_1
\end{aligned}$$

yield

$$((u \leq u_2) \wedge (u - u_3 \leq v \leq v_{\max}(u))) \vee ((u_2 \leq u) \wedge (u - u_3 \leq v \leq -u + u_3)),$$

where

$$v_{\max}(u) = \frac{uu_1 + uu_2 - 2u_1u_2 - 2uu_3 + u_1u_3 + u_2u_3}{u_1 - u_2}.$$

So that the integral can be written as

$$\left(\int_{u_1}^{u_2}\int_{u-u_3}^{v_{\max}(u)} + \int_{u_2}^{u_3}\int_{u-u_3}^{u_3-u}\right)\sqrt[4]{|K(u,0)|}\,|(u_1-u_2)(\delta_2-\delta_3)(u_3-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|\cdot$$
$$((-2)(u_1-u_3)(u_2-u_3))^{-1}\mathrm{d}v\mathrm{d}u.$$

This integral can be evaluated to be

$$\int_{u_1}^{u_2}(v_{\max}(u)-u+u_3)\sqrt[4]{|K(u,0)|}\,|(u_1-u_2)(\delta_2-\delta_3)(u_3-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|\cdot$$
$$((-2)(u_1-u_3)(u_2-u_3))^{-1}\mathrm{d}u$$
$$+\int_{u_2}^{u_3}2(u_3-u)\sqrt[4]{|K(u,0)|}\,|(u_1-u_2)(\delta_2-\delta_3)(u_3-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|\cdot$$
$$((-2)(u_1-u_3)(u_2-u_3))^{-1}\mathrm{d}u.$$

To further determine the integral we need to confront the absolute value, that is we must distinguish between $u \leq u_{\text{crit}}$ and $u \geq u_{\text{crit}}$, where

$$u_{\text{crit}} = \frac{u_1u_2\delta_1 - u_1u_3\delta_1 - u_1u_2\delta_2 + u_2u_3\delta_2 + u_1u_3\delta_3 - u_2u_3\delta_3}{u_2\delta_1 - u_3\delta_1 - u_1\delta_2 + u_3\delta_2 + u_1\delta_3 - u_2\delta_3}.$$

Finally knowing whether $(u_1-u_2)(\delta_2-\delta_3)(u_3-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)$ is positive for $u \leq u_{\text{crit}}$ or $u \geq u_{\text{crit}}$ and if $u_{\text{crit}} \in [u_1, u_2]$, $u_{\text{crit}} \in [u_2, u_3]$ or $u_{\text{crit}} \notin [u_1, u_3]$ determines the integral completely.

From the discussion above we can conclude that

$$\int\sqrt{|K|}\mathrm{d}\sigma_t = \int_{u_1}^{u_2}(v_{\max}(u)-u+u_3)(|K(u,0)|)^{3/4}\,|(u_1-u_3)(\delta_2-\delta_3)(u_2-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|\cdot$$
$$((-2)(u_1-u_3)(u_2-u_3))^{-1}\mathrm{d}u$$
$$+\int_{u_2}^{u_3}2(u_3-u)(|K(u,0)|)^{3/4}\,|(u_1-u_3)(\delta_2-\delta_3)(u_2-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|\cdot$$
$$((-2)(u_1-u_3)(u_2-u_3))^{-1}\mathrm{d}u. \tag{7}$$

Naturally the remarks regarding positivity still hold.

We remind ourselves that the proof of conjecture 7 would be complete if we could show that

$$\frac{1}{\sqrt{27}}\int_{u_1}^{u_2}(v_{\max}(u)-u+u_3)(|K(u,0)|)^{3/4}\,|(u_1-u_3)(\delta_2-\delta_3)(u_2-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|\cdot$$
$$((-2)(u_1-u_3)(u_2-u_3))^{-1}\mathrm{d}u$$
$$+\frac{1}{\sqrt{27}}\int_{u_2}^{u_3}2(u_3-u)(|K(u,0)|)^{3/4}\,|(u_1-u_3)(\delta_2-\delta_3)(u_2-u)-(u_2-u_3)(\delta_1-\delta_3)(u_1-u)|\cdot$$
$$((-2)(u_1-u_3)(u_2-u_3))^{-1}\mathrm{d}u$$
$$\leq\frac{1}{2}\max\{\sqrt[4]{|K(u_{c(13)},0)|}\left|\frac{\delta_1-\delta_3}{u_1-u_3}\right||(u_{c(13)}-u_3)(u_1-u_{c(13)})|,$$
$$(\{u_1,u_3,u_{c(13)},\delta_1,\delta_3\}\to\{u_2,u_3,u_{c(23)},\delta_2,\delta_3\}),$$
$$(\{u_1,u_3,u_{c(13)},\delta_1,\delta_3\}\to\{u_1,u_2,u_{c(12)},\delta_1,\delta_2\})\},\qquad(8)$$

where

$$u_{c(13)}=\frac{2^{1/3}\left(-2+c^2u_1u_3\right)}{\left(27c^4(u_1+u_3)+\sqrt{729c^8(u_1+u_3)^2+4\left(6c^2-3c^4u_1u_3\right)^3}\right)^{1/3}}+\frac{\left(27c^4(u_1+u_3)+\sqrt{729c^8(u_1+u_3)^2+4\left(6c^2-3c^4u_1u_3\right)^3}\right)^{1/3}}{3\cdot2^{1/3}c^2}$$

(in the appropriate domain) and the arrow denotes the replacement of variables, for example $\{\delta_1\}\to\{\delta_2\}$ means that we take the same expression as before with the replacement of δ_1 by δ_2, the notation $u_{c(13)}$ is used to emphasize the dependence of u_c on in this case u_1 and u_3.

Equation (8) can be simplified by the following observation; both sides clearly depend on only the differences $\delta_i-\delta_j$ so we may choose at least one δ_i equal to zero, because we can make use of translational symmetry. Moreover since (8) is invariant under scalar multiplication of all variables δ_i we may choose some δ_j equal to 1.

Direct verification in some simple cases

We shall verify the conjecture in special cases where we restrict the shape of the triangle so that the calculations can be performed explicitly. The cases we consider below have been chosen such that there is but one triangle between two given boundaries, it is not difficult to see that is this case the number of triangles behaves like n as the number of vertices n tends to infinity. This implies, using the same argument as in (3), it is sufficient to verify that $1/\sqrt{27}\int\sqrt{|K|}\mathrm{d}\sigma_t\leq d_o^h(\Sigma,T)$.

The first case we shall consider is the following: We assume that $u_1=0$, $u_2=u_3>0$, $\delta_1=\delta_2=0$ and $\delta_3>0$. In this case the integral $\int\sqrt{|K|}\mathrm{d}\sigma_t$ degenerates to

$$\int_0^{u_3}\frac{u(u_3-u)}{(cu^2+c^{-1})^{3/2}}\frac{\delta_3}{u_3}\mathrm{d}u=\frac{\delta_3(cu_3-\operatorname{arcsinh}(cu_3))}{c^{3/2}u_3},$$

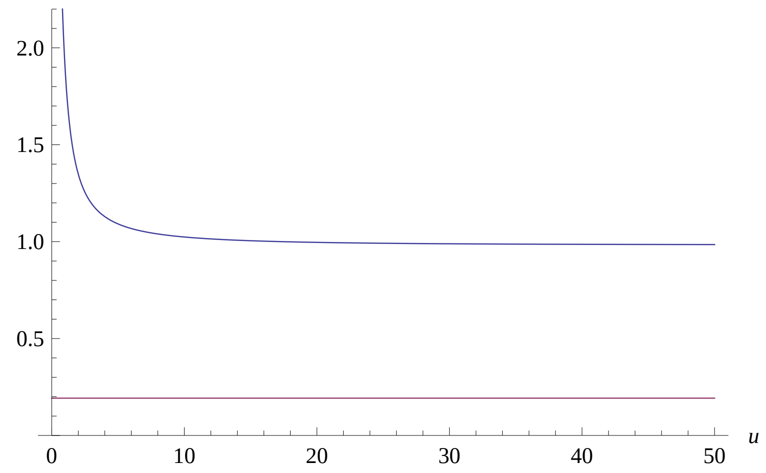

Figure 4: Depicted are $1/\sqrt{27}$ and $d_o^h(t,\Sigma)/\int\sqrt{|K|}\mathrm{d}\sigma_t$ for $u_1 = 0$, $u_2 = u_3 > 0$ and $\delta_1 = \delta_2 = 0$.

which can be deduced from (6). While the one-sided Hausdorff distance is given by

$$d_o^h(t,\Sigma) = \frac{\delta_3}{u_3}\frac{u_c(u_3-u_c)}{\sqrt{cu_c^2+c^{-1}}},$$

where

$$u_c = \frac{-4\cdot 3^{1/3}c^2 + 2^{1/3}\left(9c^4u_3+\sqrt{2^5 3c^6+3^4c^8u_3^2}\right)^{2/3}}{6^{2/3}c^2\left(9c^4u_3+\sqrt{2^5 3c^6+3^4c^8u_3^2}\right)^{1/3}}.$$

If we now further assume that $c = 1$, the desired inequality, the equivalent of (8) in this degenerate case, is obvious from a plot, see figure 4.

The following case we study are triangles of which one side lies on a ruling, that is $\delta_1 = \delta_2 = 0$ and $-u_1 = u_2 = u_3 = \tilde{u} > 0$. This includes the triangulation of a hyperboloid bounded by two congruent circles, in the manner suggested by Fejes Tóth. This case is very easy because we have due to symmetry that $u_c = 0$ and thus the Hausdorff distance equals

$$d_o^h(t,\Sigma) = \frac{1}{2}\sqrt{c}\delta_3\tilde{u},$$

while the integral part degenerates into

$$\int\sqrt{|K(u,0)|}\mathrm{d}\sigma_t = \int_0^1\int_0^{1-\lambda_1}|K(u,0)|^{3/4}|2\tilde{u}\delta_3(\tilde{u}-u)|\mathrm{d}\lambda_2\mathrm{d}\lambda_1,$$

where $u = \tilde{u} - 2\tilde{u}\lambda_1$. So that we are faced with

$$\int_0^1\frac{4\tilde{u}\delta_3\lambda(1-\lambda)}{(\tilde{u}^2(1-2\lambda)^2c+c^{-1})^{3/2}}\mathrm{d}\lambda = \frac{\delta_3(c\,\tilde{u}\sqrt{1+c^2\tilde{u}^2}-\operatorname{arcsinh}(c\,\tilde{u}))}{c^{3/2}\tilde{u}}.$$

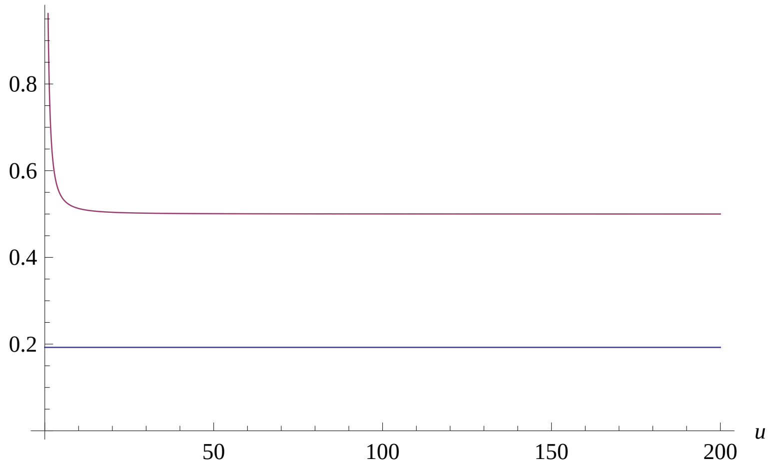

Figure 5: Depicted are $1/\sqrt{27}$ and $d_o^h(t,\Sigma)/\int\sqrt{|K|}\mathrm{d}\sigma_t$ for $\delta_1=\delta_2=0$ and $-u_1=u_2=u_3=\tilde{u}>0$.

If we choose $c=1$ we can verify this inequality in more or less the same manner as the previous case discussed, see figure 5.

In this case we are even able to prove the statement for any c. Consider

$$\frac{d_o^h(t,\Sigma)}{\int\sqrt{|K|}\mathrm{d}\sigma_t}=\frac{c^2\tilde{u}^2}{2c\tilde{u}\sqrt{1+c^2\tilde{u}^2}-2\mathrm{arcsinh}(c\tilde{u})}$$

and its derivative

$$\frac{\partial}{\partial\tilde{u}}\frac{d_o^h(t,\Sigma)}{\int\sqrt{|K|}\mathrm{d}\sigma_t}=\frac{c^2\tilde{u}\left(c\tilde{u}-\sqrt{1+c^2\tilde{u}^2}\mathrm{arcsinh}(c\tilde{u})\right)}{\sqrt{1+c^2\tilde{u}^2}\left(c\tilde{u}\sqrt{1+c^2\tilde{u}^2}-\mathrm{arcsinh}(c\tilde{u})\right)^2}.$$

It is clear that the derivative is unequal to zero for all $\tilde{u}>0$ so the maximum is not attained on $[0,\infty)$ on the other hand we have that

$$\lim_{\tilde{u}\to\infty}\frac{c^2\tilde{u}\left(c\tilde{u}-\sqrt{1+c^2\tilde{u}^2}\mathrm{arcsinh}(c\tilde{u})\right)}{\sqrt{1+c^2\tilde{u}^2}\left(c\tilde{u}\sqrt{1+c^2\tilde{u}^2}-\mathrm{arcsinh}(c\tilde{u})\right)^2}=\frac{1}{2}>\frac{1}{\sqrt{27}}.$$

A On the approximation of C^2 convex surfaces by C^3 convex surfaces

In this section we prove the the following corollary:

Corollary 1 *Let M be a C^2 convex surface, that is, M is the boundary of a convex set S_C, in $\mathbb{R}^n$ with $n \geq 3$. We further assume that M is compact, has nowhere zero Gaussian curvature and the interior of the set S is nonempty. Then for each $\varepsilon > 0$ there exists a C^∞ convex surface M_ε such that $d^H(M, M_\varepsilon) < \varepsilon$.*

The above corollary is a consequence of the following theorem within the field of differential topology:

Theorem 2 *Let N_1 and N_2 be C^s manifolds, $1 \leq s \leq \infty$. Then $C^s(N_1, N_2)$ is dense in $C^r_S(N_1, N_2)$, $0 \leq r < s$.*

Here $C^r_S(N_1, N_2)$ denotes the set of the C^r maps from N_1 to N_2 endowed with the strong topology. This theorem can be found as theorem 2.6 in [12]. Our corollary was inspired by the efforts of Peter Gruber in [8] to extend the results of Rolf Schneider [16] for C^3 convex surfaces to C^2 convex surface.

Proof Let M be as described in Corollary 1. We start by choosing a point $p \in \mathrm{Int}(S_C)$. Because S_C is convex we have that for every point $m \in M = \partial S_C$ the line connecting p and m lies in S_C. Due to translation we may assume that p is the origin. We therefore may parameterize M by a C^2 function $r : S^{n-1} \to \mathbb{R}$ so that $r(v)v \in M$ for all $v \in S^{n-1}$ and for every $m \in M$ there is a unique $v \in S^{n-1}$ such that $r(v)v = m$. It follows directly from definition

$$\mathrm{II}(p)(v, w) = -\langle \mathrm{d}\nu(v), w\rangle,$$

where $\mathrm{d}\nu$ denotes the Weingarten map, that the second fundamental form of M using this parametrization depends continuously on r and its first and second order derivatives. Theorem 2, with $N_1 = S^{n-1}$ and $N_2 = \mathbb{R}$, provides us with a C^∞ map r_ε such that $|D^k(r_\varepsilon\phi^{-1})(x) - D^k(r\phi^{-1})(x)| < \varepsilon$, where ϕ is a chart of S^{n-1} and $k = 0, 1, 2$. We shall now prove that $M_\varepsilon = \{r_\varepsilon(v)v \in \mathbb{R}^n | v \in S^{n-1}\}$ is a convex surface with nowhere zero Gaussian curvature, provided that ε is sufficiently small. We have demanded that M has nowhere zero Gaussian curvature, this implies that the second fundamental form of M is positive definite. Since the second fundamental form is continuous in r and its first and second order derivatives we see that, for ε sufficiently smal,l the second fundamental form of M_ε, now depending on r_ε and its first and second order derivatives, is still positive definite and therefore the Gaussian curvature of M_ε is nowhere zero. Thanks to Hadamard, page 352-353 of [10] (see also page 1048-1049 of [9]), we have that:

Theorem 3 *Let $x : N \to \mathbb{R}^n$ $(n \geq 3)$ be a compact oriented hypersurface M of class C^2 with positive Gaussian curvature. Then x is an embedding and $x(N)$ is equal to the boundary of a suitable compact convex body S_C in $\mathbb{R}^n$.*

This implies that M_ε is indeed a convex surface. Finally we see that for every point $m = r(v)v \in M$ there is a point $m_\varepsilon = r_\varepsilon(v)v \in M_\varepsilon$ which lies at most ε from m. This yields that the Hausdorff distance between M and M_ε is less then ε. □

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