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Computational Geometric Learning

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CGL Technical Report No.: 36

Part of deliverable: WP-WP1/
Site: INRIA Saclay
Month: 24

Project co-funded by the European Commission within FP7 (2010–2013)
under contract nr. IST-25582

Persistence stability for geometric complexes

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July 16, 2012

Abstract

In this paper we study the properties of the homology of different geometric filtered complexes (such as Vietoris–Rips, Čech and witness complexes) built on top of precompact spaces. Using recent developments in the theory of topological persistence [6] we provide simple and natural proofs of the stability of the persistent homology of such complexes with respect to the Gromov–Hausdorff distance. We also exhibit a few noteworthy properties of the homology of the Rips and Čech complexes built on top of compact spaces.

1 Introduction

Inferring topological properties of metric spaces from approximations is a problem that has attracted special attention in computational topology recently. Given a metric space (Y, d_Y) approximating an unknown metric space (X, d_X) , the aim is to build a simplicial complex on the vertex set Y whose homology or homotopy type is the same as X . Note that, although Y is finite in many applications, finiteness is not a requirement a priori.

Among the many geometric simplicial complexes available to us, the Vietoris–Rips complex (or simply ‘Rips complex’) is particularly useful, being easy to compute and having good approximation properties. We recall the definition. Let (X, d_X) be a metric space and α a real parameter (the ‘scale’). Then $\text{Rips}(X, \alpha)$ is the simplicial complex on X whose simplices are the finite subsets of X with diameter at most α :

$$\sigma = \{x_0, x_1, \dots, x_k\} \in \text{Rips}(X, \alpha) \Leftrightarrow d_X(x_i, x_j) \leq \alpha \quad \text{for all } i, j \in \{0, \dots, k\}$$

When (X, d_X) is a closed Riemannian manifold, J.-C. Hausmann [14] proved that if $\alpha > 0$ is sufficiently small then the geometric realisation of $\text{Rips}(X, \alpha)$ is homotopy equivalent to X . This result was later generalised by J. Latschev [15], who proved that if (Y, d_Y) is sufficiently close

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to (X, d_X) in the Gromov–Hausdorff distance, then there exists $\alpha > 0$ such that $\text{Rips}(Y, \alpha)$ is homotopy equivalent to X . Recently, Attali et al. [1] adapted these results to a class of sufficiently regular compact subsets of Euclidean spaces. For larger classes of compact subsets of Riemannian manifolds, the homology and homotopy of such sets are known to be encoded in nested pairs of Vietoris–Rips complexes [7], yet it remains still open whether or not a single Rips complex can carry this topological information.

These approaches make it possible to recover the topology of a metric space (X, d_X) from a sufficiently close approximation (Y, d_Y) , provided that the parameter α is chosen correctly. Unfortunately, this choice very much depends on the geometry of X and can be difficult (if even possible) to determine in practical applications. One way round this issue is to use topological persistence [10, 16], which encodes the homology of the entire nested family $\mathbb{R}\text{ips}(X) = (\text{Rips}(X, \alpha))_{\alpha \in \mathbf{R}}$ in a single invariant, the persistence diagram. Relevant scales α can then be selected by the user, and the diagram provides an explicit relationship between the choice of a scale α and the homology of the corresponding Vietoris–Rips complex.

The stability of this construction was established by Chazal et al. [4], who proved that

$$d_b(\text{dgm}(\mathbb{R}\text{ips}(X)), \text{dgm}(\mathbb{R}\text{ips}(Y))) \leq 2 d_{\text{GH}}(X, Y), \quad (1.1)$$

for finite metric spaces (X, d_X) and (Y, d_Y) . Here d_b and d_{GH} denote the bottleneck [9, Chap. 8] and Gromov–Hausdorff distances, respectively. The bound turns out to be tight, which motivates the use of persistence diagrams as discriminative signatures to compare geometric shapes represented as finite metric spaces.

In this paper we show that the same inequality holds for all precompact metric spaces, and can in fact be extended to a larger class of filtered geometric complexes. Our analysis adopts a new perspective, based on recent developments in the theory of topological persistence [6], which results in simple and natural proofs. Our contributions are the following:

- Extending the concept of simplicial map between complexes to the one of ε -simplicial multivalued map between filtered complexes, we show that such maps induce canonical ε -interleavings between the persistent homology modules of these complexes — Section 3.
- Applying this result to correspondences between metric spaces, we prove the ε -interleaving of the persistent homology modules of certain families of filtered geometric complexes (including Rips filtrations) built on top of ε -close metric spaces — Section 4.
- We prove the tameness of the persistent homology modules of the above filtered complexes when the vertex sets are precompact. Combined with the previous results, this result shows that inequality 1.1 can be generalised as claimed above — Section 5.1.

In addition to this consistent set of results, the second part of Section 5 presents a few noteworthy properties of the homology groups of Rips and Čech complexes of precompact metric spaces.

2 Persistence modules and persistence diagrams

We adopt the approach and the notation of [6]. In this section we recall the definitions and results that we need. For a detailed presentation the reader is referred to [6].

A persistence module \mathbb{V} over the real numbers \mathbf{R} is an indexed family of vector spaces¹ $(V_a \mid a \in \mathbf{R})$ and a doubly-indexed family of linear maps $(v_a^b : V_a \rightarrow V_b \mid a \leq b)$ which satisfy the composition law $v_b^c \circ v_a^b = v_a^c$ whenever $a \leq b \leq c$, and where v_a^a is the identity map on V_a .

Example 2.1 (homology of a filtered complex). This is the standard example, which we use throughout this paper. Let \mathbb{S} be a filtered simplicial complex: that is, a family $(\mathbb{S}_a \mid a \in \mathbf{R})$ of subcomplexes of some fixed simplicial complex $\bar{\mathbb{S}}$, such that $\mathbb{S}_a \subseteq \mathbb{S}_b$ whenever $a \leq b$. Let $V_a = H(\mathbb{S}_a)$ be the homology group² of \mathbb{S}_a , and let $v_a^b : H(\mathbb{S}_a) \rightarrow H(\mathbb{S}_b)$ be the linear map induced by the inclusion $\mathbb{S}_a \hookrightarrow \mathbb{S}_b$. Since, for any $a \leq b \leq c$, the inclusion $\mathbb{S}_a \hookrightarrow \mathbb{S}_c$ is the composition of the inclusions $\mathbb{S}_a \hookrightarrow \mathbb{S}_b$ and $\mathbb{S}_b \hookrightarrow \mathbb{S}_c$, the linear maps satisfy $v_a^c = v_b^c \circ v_a^b$ and the family $(H(\mathbb{S}_a) \mid a \in \mathbf{R})$ is a persistence module.

Let \mathbb{U}, \mathbb{V} be persistence modules over \mathbf{R} , and let ϵ be any real number. A **homomorphism of degree ϵ** is a collection Φ of linear maps

$$(\phi_a : U_a \rightarrow V_{a+\epsilon} \mid a \in \mathbf{R})$$

such that $v_{a+\epsilon}^{b+\epsilon} \circ \phi_a = \phi_b \circ v_a^b$ for all $a \leq b$. We write

$$\begin{aligned} \text{Hom}^\epsilon(\mathbb{U}, \mathbb{V}) &= \{\text{homomorphisms } \mathbb{U} \rightarrow \mathbb{V} \text{ of degree } \epsilon\}, \\ \text{End}^\epsilon(\mathbb{V}) &= \{\text{homomorphisms } \mathbb{V} \rightarrow \mathbb{V} \text{ of degree } \epsilon\}. \end{aligned}$$

Composition is defined in the obvious way.

For $\epsilon \geq 0$, the most important degree- ϵ endomorphism is the shift map

$$1_{\mathbb{V}}^\epsilon \in \text{End}^\epsilon(\mathbb{V}),$$

which is the collection of maps $(v_a^{a+\epsilon})$ from the persistence structure on \mathbb{V} . If Φ is a homomorphism $\mathbb{U} \rightarrow \mathbb{V}$ of any degree, then by definition $\Phi 1_{\mathbb{U}}^\epsilon = 1_{\mathbb{V}}^\epsilon \Phi$ for all $\epsilon \geq 0$.

Example (2.1 continued). Given $\epsilon > 0$, if $f : \bar{\mathbb{S}} \rightarrow \bar{\mathbb{S}}'$ is a simplicial map such that f maps \mathbb{S}_a to $\mathbb{S}'_{a+\epsilon}$ for any $a \in \mathbf{R}$, then f induces a homomorphism of degree ϵ between the persistence modules $H(\mathbb{S})$ and $H(\mathbb{S}')$.

Two persistence modules \mathbb{U}, \mathbb{V} are said to be **ϵ -interleaved** if there are maps

$$\Phi \in \text{Hom}^\epsilon(\mathbb{U}, \mathbb{V}), \quad \Psi \in \text{Hom}^\epsilon(\mathbb{V}, \mathbb{U})$$

such that $\Psi\Phi = 1_{\mathbb{U}}^{2\epsilon}$ and $\Phi\Psi = 1_{\mathbb{V}}^{2\epsilon}$.

¹All vector spaces are taken to be over an arbitrary field \mathbf{k} , fixed throughout this paper.

²We use simplicial homology with coefficients in the field \mathbf{k} .

Following [4, 6] we say that a persistence module \mathbb{V} is **q-tame** if

$$r_a^b = \text{rank}(v_a^b) < \infty \quad \text{whenever } a < b.$$

This regularity condition ensures that persistence modules behave well:

Theorem 2.2 ([6], Section 2.8 and Theorem 4.21). *If \mathbb{U} is a q-tame module then it has a well-defined persistence diagram $\text{dgm}(\mathbb{U})$. If \mathbb{U}, \mathbb{V} are q-tame persistence modules that are ϵ -interleaved then there exists a ϵ -matching between the multisets $\text{dgm}(\mathbb{U}), \text{dgm}(\mathbb{V})$, i.e. the bottleneck distance between the diagrams is bounded: $d_b(\text{dgm}(\mathbb{U}), \text{dgm}(\mathbb{V})) \leq \epsilon$.*

3 Multivalued maps

The notion of a simplicial map between simplicial complexes extends to the notion of an ϵ -simplicial map between filtered simplicial complexes in the following way:

Definition 3.1. Let \mathbb{S} and \mathbb{T} be two filtered simplicial complexes with vertex sets X and Y respectively. A map $f : X \rightarrow Y$ is ϵ -simplicial from \mathbb{S} to \mathbb{T} if it induces a simplicial map $\mathbb{S}_a \rightarrow \mathbb{T}_{a+\epsilon}$ for every $a \in \mathbf{R}$. Equivalently, f is ϵ -simplicial if and only if for any $a \in \mathbf{R}$ and any simplex $\sigma \in \mathbb{S}_a$, $f(\sigma)$ is a simplex of $\mathbb{T}_{a+\epsilon}$.

We wish to extend this concept to multivalued maps. Here are the basic notions.

A **multivalued map** $C : X \rightrightarrows Y$ from a set X to a set Y is a subset of $X \times Y$, also denoted C , that projects surjectively onto X through the canonical projection $\pi_X : X \times Y \rightarrow X$. The image $C(\sigma)$ of a subset σ of X is the canonical projection onto Y of the preimage of σ through π_X .

A (single-valued) map f from X to Y is **subordinate** to C if we have $(x, f(x)) \in C$ for every $x \in X$. In that case we write $f : X \xrightarrow{C} Y$.

The **composite** of two multivalued maps $C : X \rightrightarrows Y$ and $D : Y \rightrightarrows Z$ is the multivalued map $D \circ C : X \rightrightarrows Z$, defined by:

$$(x, z) \in D \circ C \Leftrightarrow \text{there exists } y \in Y \text{ such that } (x, y) \in C \text{ and } (y, z) \in D$$

The **transpose** of C , denoted C^T , is the image of C through the symmetry map $(x, y) \mapsto (y, x)$. Although C^T is well-defined as a subset of $Y \times X$, it is not always a multivalued map because it may not project surjectively onto Y .

We now discuss simplicial multivalued maps.

Definition 3.2. Let \mathbb{S} and \mathbb{T} be two filtered simplicial complexes with vertex sets X and Y respectively. A multivalued map $C : X \rightrightarrows Y$ is ϵ -simplicial from \mathbb{S} to \mathbb{T} if for any $a \in \mathbf{R}$ and any simplex $\sigma \in \mathbb{S}_a$, every finite subset of $C(\sigma)$ is a simplex of $\mathbb{T}_{a+\epsilon}$.

Proposition 3.3. *Let $C : X \rightrightarrows Y$ be an ϵ -simplicial multivalued map from \mathbb{S} to \mathbb{T} . Then C induces a canonical linear map $\mathbf{H}(C) \in \text{Hom}^\epsilon(\mathbf{H}(\mathbb{S}), \mathbf{H}(\mathbb{T}))$, equal to $\mathbf{H}(f)$ for any f subordinate to C .*

Proof. Any choice of f induces a simplicial map $\mathbb{S}_a \rightarrow \mathbb{T}_{a+\epsilon}$ at each $a \in \mathbf{R}$, and these maps commute with the inclusions $\mathbb{S}_a \hookrightarrow \mathbb{S}_b, \mathbb{T}_{a+\epsilon} \hookrightarrow \mathbb{T}_{b+\epsilon}$ for all $a \leq b$. Thus f induces $H(f) \in \text{Hom}^\epsilon(H(\mathbb{S}), H(\mathbb{T}))$. Any two subordinate maps $f_1, f_2 : X \xrightarrow{C} Y$ induce simplicial maps $\mathbb{S}_a \rightarrow \mathbb{T}_{a+\epsilon}$ which are contiguous and therefore $H(f_1) = H(f_2)$. Thus the map $H(C)$ is uniquely defined. \square

Another immediate consequence is that the induced homomorphism is invariant under taking subsets of C that are also multivalued maps:

Proposition 3.4. *If $C' \subseteq C : X \rightrightarrows Y$ and C is ϵ -simplicial from \mathbb{S} to \mathbb{T} , then C' is ϵ -simplicial from \mathbb{S} to \mathbb{T} and $H(C') = H(C)$*

Proof. Since C' is a multivalued map contained in C , it is also ϵ -simplicial, and any map $f : X \rightarrow Y$ that is subordinate to C' is also subordinate to C , so we have $H(C') = H(C)$. \square

Finally, induced homomorphisms compose in the natural way:

Proposition 3.5. *Let $\mathbb{S}, \mathbb{T}, \mathbb{U}$ be filtered complexes with vertex sets X, Y, Z respectively. If*

$$\begin{aligned} C : X \rightrightarrows Y & \text{ is a } \epsilon\text{-simplicial multivalued map from } \mathbb{S} \text{ to } \mathbb{T}, \\ D : Y \rightrightarrows Z & \text{ is a } \delta\text{-simplicial multivalued map from } \mathbb{T} \text{ to } \mathbb{U}, \end{aligned}$$

then the composite $D \circ C : X \rightrightarrows Z$ is a $(\epsilon + \delta)$ -simplicial multivalued map from \mathbb{S} to \mathbb{U} , and $H(D \circ C) = H(D) \circ H(C)$.

Proof. $D \circ C$ is $(\epsilon + \delta)$ -simplicial as an immediate consequence of the definition of ϵ -simplicial multivalued map. Let $f : X \xrightarrow{C} Y$ be subordinate to C , and let $g : Y \xrightarrow{D} Z$ be subordinate to D . The composite $g \circ f : X \xrightarrow{D \circ C} Z$ is subordinate to $D \circ C$, therefore $H(D \circ C) = H(D) \circ H(C)$. \square

4 Correspondences

4.1 Interleaving persistence modules of filtered complexes through correspondences

Definition 4.1. A multivalued map $C : X \rightrightarrows Y$ is a **correspondence** if the canonical projection $C \rightarrow Y$ is surjective, or equivalently, if C^T is also a multivalued map.

We immediately deduce, if C is a correspondence, that the identity maps $\mathbb{1}_X = \{(x, x) : x \in X\}$ and $\mathbb{1}_Y = \{(y, y) : y \in Y\}$ satisfy:

$$\begin{aligned} \mathbb{1}_X & \subseteq C^T \circ C \\ \mathbb{1}_Y & \subseteq C \circ C^T \end{aligned}$$

From this property and propositions 3.4 and 3.5, we deduce the following result:

Proposition 4.2. *Let \mathbb{S}, \mathbb{T} be filtered complexes with vertex sets X, Y respectively. If $C : X \rightrightarrows Y$ is a correspondence such that C and C^T are both ε -simplicial, then together they induce a canonical ε -interleaving between $H(\mathbb{S})$ and $H(\mathbb{T})$, the interleaving homomorphisms being $H(C)$ and $H(C^T)$.*

4.2 Applications to filtered complexes on metric spaces

When (X, d_X) and (Y, d_Y) are metric spaces, the distortion of a correspondence $C : X \rightrightarrows Y$ is defined as follows:

$$\text{dis}(C) = \sup\{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in C\}$$

The Gromov–Hausdorff distance ([3], Theorem 7.3.25) between (X, d_X) and (Y, d_Y) is then defined as the infimum of the distortions among all the correspondences between X and Y :

$$d_{\text{GH}}(X, Y) = \frac{1}{2} \inf\{\text{dis}(C) : C \text{ is a correspondence } X \rightrightarrows Y\}$$

Although d_{GH} is not necessarily finite, it is a distance on the set of isometry classes of compact metric spaces:

- it is zero if and only if the spaces are isometric;
- a correspondence and its transpose have the same distortion, so d_{GH} is symmetric;
- the composite of two correspondences C, C' is a correspondence and $\text{dis}(C' \circ C) \leq \text{dis}(C') + \text{dis}(C)$ so d_{GH} satisfies the triangle inequality: $d_{\text{GH}}(X, Z) \leq d_{\text{GH}}(X, Y) + d_{\text{GH}}(Y, Z)$.

The theme of the next few examples is that low-distortion correspondences give rise to ε -simplicial maps on filtered complexes.

The Čech complex

Let (X, d_X) be a metric space. For $a \in \mathbf{R}$ we define a simplicial complex $\check{\text{Cech}}(X, a)$ on the vertex set X by the following condition:

$$[x_0, x_1, \dots, x_k] \in \check{\text{Cech}}(X, a) \iff \bigcap_{i=0}^k B(x_i, a) \neq \emptyset$$

Here $B(x, a) = \{x' \in X : d_X(x, x') \leq a\}$ denotes the closed ball with centre $x \in X$ and radius a . Any point \bar{x} in the intersection $\bigcap_i B(x_i, a)$ is called an a -**centre** for the simplex $[x_0, \dots, x_k]$.

For $a \leq 0$, note that $\check{\text{Cech}}(X, a)$ consists of the vertex set X alone. There is a natural inclusion $\check{\text{Cech}}(X, a) \subseteq \check{\text{Cech}}(X, b)$ whenever $a \leq b$. Thus, the simplicial complexes $\check{\text{Cech}}(X, a)$ together with these inclusion maps define a filtered simplicial complex $\check{\text{Cech}}(X)$ on X .

Lemma 4.3. *Let $(X, d_X), (Y, d_Y)$ be metric spaces. For any $\varepsilon > 2d_{\text{GH}}(X, Y)$ the persistence modules $H(\check{\text{Cech}}(X))$ and $H(\check{\text{Cech}}(Y))$ are ε -interleaved.*

Proof. Let $C : X \rightrightarrows Y$ be a correspondence with distortion at most ε .

Consider $\sigma \in \check{\text{Cech}}(X, a)$. Let \bar{x} be an a -centre for σ , so $d_X(\bar{x}, x) \leq a$ for all $x \in \sigma$. Pick $\bar{y} \in C(\bar{x})$. Now for any $y \in C(\sigma)$ we have $y \in C(x)$ for some $x \in \sigma$, and therefore:

$$d_Y(\bar{y}, y) \leq d_X(\bar{x}, x) + \varepsilon \leq a + \varepsilon$$

Let $\tau \subseteq C(\sigma)$ be any finite subset; then \bar{y} is an $(a + \varepsilon)$ -centre for τ and hence $\tau \in \check{\text{Cech}}(Y, a + \varepsilon)$.

We have shown that C is ε -simplicial from $\check{\text{Cech}}(X)$ to $\check{\text{Cech}}(Y)$. Symmetrically, C^T is ε -simplicial from $\check{\text{Cech}}(Y)$ to $\check{\text{Cech}}(X)$. The result now follows from Proposition 4.2. \square

The Vietoris–Rips complex

Let (X, d_X) be a metric space. For $a \in \mathbf{R}$ we define a simplicial complex $\text{Rips}(X, a)$ on the vertex set X by the following condition:

$$[x_0, x_1, \dots, x_k] \in \text{Rips}(X, a) \iff d_X(x_i, x_j) \leq a \text{ for all } i, j$$

For $a \leq 0$, note that $\text{Rips}(X, a)$ consists of the vertex set X alone. There is a natural inclusion $\text{Rips}(X, a) \subseteq \text{Rips}(X, b)$ whenever $a \leq b$. Thus, the simplicial complexes $\text{Rips}(X, a)$ together with these inclusion maps define a filtered simplicial complex $\mathbb{R}\text{ips}(X)$ on X .

Lemma 4.4. *Let (X, d_X) , (Y, d_Y) be metric spaces. For any $\varepsilon > 2\text{d}_{\text{GH}}(X, Y)$ the persistence modules $\text{H}(\mathbb{R}\text{ips}(X))$ and $\text{H}(\mathbb{R}\text{ips}(Y))$ are ε -interleaved.*

Proof. Let $C : X \rightrightarrows Y$ be a correspondence with distortion at most ε .

If $\sigma \in \text{Rips}(X, a)$ then $d_X(x, x') \leq a$ for all $x, x' \in \sigma$. Let $\tau \subseteq C(\sigma)$ be any finite subset. For any $y, y' \in \tau$ there exist $x, x' \in \sigma$ such that $y \in C(x)$, $y' \in C(x')$, and therefore:

$$d_Y(y, y') \leq d_X(x, x') \leq a + \varepsilon$$

It follows that $\tau \in \text{Rips}(Y, a + \varepsilon)$.

We have shown that C is ε -simplicial from $\mathbb{R}\text{ips}(X)$ to $\mathbb{R}\text{ips}(Y)$. Symmetrically, C^T is ε -simplicial from $\mathbb{R}\text{ips}(Y)$ to $\mathbb{R}\text{ips}(X)$. The result now follows from Proposition 4.2. \square

The witness complex

Let L, W be two sets and $\Lambda : W \times L \rightarrow \mathbf{R}_+$ a non negative function. For $a \in \mathbf{R}$ we define a simplicial complex $\text{Wit}(L, W; a)$ with vertex set L by the following conditions:

$$\sigma \in \text{Wit}(L, W; a) \iff \begin{cases} \forall \tau \subsetneq \sigma, \tau \in \text{Wit}(L, W; a) \\ \text{and} \\ \exists w \in W \text{ s.t. } \Lambda(w, l) \leq \Lambda(w, l') + a \text{ for all } l \in \sigma \text{ and } l' \in L \setminus \sigma \end{cases}$$

Whenever the second condition is verified by a simplex $\sigma \subset L$ and a point $w \in W$, we say w is an *a-witness* for σ . An *a-witness* is obviously a *b-witness* for any $b \geq a$, so there is a natural inclusion $\text{Wit}(L, W; a) \subseteq \text{Wit}(L, W; b)$. The simplicial complexes $\text{Wit}(L, W; a)$ together with these inclusion maps define a filtered simplicial complex $\mathbb{W}\text{it}(L, W)$ with vertex set L .

Let L be a set, W, W' be two witness sets of L with respect to maps $\Lambda : W \times L \rightarrow \mathbf{R}_+$ and $\Lambda' : W' \times L \rightarrow \mathbf{R}_+$. The distortion (with respect to L, Λ and Λ') of a correspondence $C : W \rightrightarrows W'$ is defined by

$$\text{dis}(C) = \sup_{(w, w') \in C} \sup_{l \in L} |\Lambda(w, l) - \Lambda'(w', l)|$$

Lemma 4.5. *Let L be a set, W, W' be two witness sets of L with respect to maps $\Lambda : W \times L \rightarrow \mathbf{R}_+$ and $\Lambda' : W' \times L \rightarrow \mathbf{R}_+$. For any ε such that*

$$\varepsilon > \inf\{\text{dis}(C) : C \text{ is a correspondence } W \rightrightarrows W'\}$$

the persistence modules $\text{H}(\mathbb{W}\text{it}(L, W))$ and $\text{H}(\mathbb{W}\text{it}(L, W'))$ are ε -interleaved.

Proof. Let $C : W \rightrightarrows W'$ be a correspondence such that $\text{dis}(C) < \varepsilon$. We claim that every simplex $\sigma \subseteq L$ with an *a-witness* in W has an $(a + \varepsilon)$ -witness in W' . Indeed, let $w \in W$ be an *a-witness* for σ , and let $w' \in W'$ be such that $(w, w') \in C$. Then, for all $l \in \sigma$ and $l' \in L \setminus \sigma$, we have $\Lambda'(w', l) \leq \Lambda(w, l) + \varepsilon \leq \Lambda(w, l') + a + \varepsilon$, and so w' is an $(a + \varepsilon)$ -witness for σ .

It follows from our claim that the identity map $\mathbb{1}_L$ is ε -simplicial from $\mathbb{W}\text{it}(L, W)$ to $\mathbb{W}\text{it}(L, W')$. Symmetrically, $\mathbb{1}_L^T = \mathbb{1}_L$ is ε -simplicial from $\mathbb{W}\text{it}(L, W')$ to $\mathbb{W}\text{it}(L, W)$, and we conclude as in Lemmas 4.3 and 4.4 that $\text{H}(\mathbb{W}\text{it}(L, W))$ and $\text{H}(\mathbb{W}\text{it}(L, W'))$ are ε -interleaved. \square

Example 4.6. If (X, d_X) is a metric space, and L, W are subsets of X , Λ can be chosen as the restriction of d_X to $W \times L$: $\Lambda(w, x) = d_X(w, x)$ for any $(w, x) \in W \times L$. Then Lemma 4.5 can be restated in the following way.

Lemma 4.7. *Let (X, d_X) be a metric space, and let L, W, W' be subsets of X . For any ε greater than the Hausdorff distance $d_{\text{H}}(W, W')$ the persistence modules $\text{H}(\mathbb{W}\text{it}(L, W))$ and $\text{H}(\mathbb{W}\text{it}(L, W'))$ are ε -interleaved.*

Proof. Since $\varepsilon > d_{\text{H}}(W, W')$, $C = \{(w, w') \in W \times W' : d_X(w, w') < \varepsilon\}$ is a correspondence between W and W' . Moreover for any $l \in L$ and $(w, w') \in C$, $|d_X(w, l) - d_X(w', l)| \leq d_X(w, w') < \varepsilon$. So the distortion of C with respect to L is upper bounded by ε . \square

Note that there is no equivalent of Lemma 4.5 in the case where the set L is perturbed, even if the set of witnesses is constrained to stay fixed ($W = W'$). Here is a counterexample: on the real line, consider the sets $W = W' = L = \{0, 1\}$ and $L' = \{-\delta, 0, 1, 1 + \delta\}$, where $\delta \in (0, 1/2)$ is arbitrary. Then, $\text{Wit}(L, W; a) = \{[0], [1], [0, 1]\}$ for all $a \geq 0$, while $\text{Wit}(L', W; a) = \{[-\delta], [0], [1], [1 + \delta], [-\delta, 0], [1, 1 + \delta]\}$ for all $a \in [\delta, 1 - \delta)$. Thus, $\text{H}(\mathbb{W}\text{it}(L, W))$ and $\text{H}(\mathbb{W}\text{it}(L', W))$ are not ε -interleaved for any value $\varepsilon < 1 - 2\delta$, while $d_{\text{H}}(L, L') = \delta$ can be made arbitrarily small compared to $1 - 2\delta$.

Generalisation to dissimilarity spaces

In data analysis one often considers data sets X equipped with a dissimilarity measure, i.e. a map $\tilde{d}_X : X \times X \rightarrow \mathbf{R}$ that satisfies $\tilde{d}_X(x, x) \leq \tilde{d}_X(x, y) = \tilde{d}_X(y, x)$ for all $x, y \in X$ but is not required to satisfy any of the other metric space axioms. It is easily seen that Čech, Vietoris–Rips and witness complexes can still be defined over such a space X , and that the distortion of a correspondence $C : X \rightrightarrows Y$ between such spaces is well-defined. Moreover, since the proofs of our interleaving results do not make use of any other distance axiom (triangle inequality, non-negativity, zero property), they remain valid for such spaces.

5 Regularity of Rips and Čech filtrations

5.1 Stability of persistence for Rips and Čech filtrations on precompact metric spaces

Given a positive real number $\varepsilon > 0$, a subset $F \subseteq X$ of a metric space (X, d_X) is an ε -**sample** of X if for any $x \in X$ there exists $f \in F$ such that $d_X(x, f) < \varepsilon$. A precompact metric space (X, d_X) is a space such that X has a finite ε -sample for every $\varepsilon > 0$.

Proposition 5.1. *If (X, d_X) is a precompact metric space then the persistence modules $H(\check{\text{Cech}}(X))$ and $H(\mathbb{R}\text{ips}(X))$ are q -tame.*

Proof. Let us first consider the case of the Vietoris–Rips persistence module. We must show that the map $I_a^b : H(\mathbb{R}\text{ips}(X, a)) \rightarrow H(\mathbb{R}\text{ips}(X, b))$ induced by the inclusion has finite rank whenever $a < b$. Let $\epsilon = (b - a)/2$. Since X is precompact there exists a finite $\varepsilon/2$ -sample F of X . The set $C = \{(x, f) \in X \times F : d_X(x, f) < \varepsilon/2\}$ is an ε -correspondence so the Gromov–Hausdorff distance between F and X is upper bounded by $\varepsilon/2$. It then follows from Lemma 4.4 that there exists an ε -interleaving between $H(\mathbb{R}\text{ips}(X))$ and $H(\mathbb{R}\text{ips}(F))$. Using the interleaving maps, I_a^b factorises as

$$H(\mathbb{R}\text{ips}(X, a)) \rightarrow H(\mathbb{R}\text{ips}(F, a + \epsilon)) \rightarrow H(\mathbb{R}\text{ips}(X, a + 2\epsilon)) = H(\mathbb{R}\text{ips}(X, b)).$$

The second term is finite dimensional since $\mathbb{R}\text{ips}(F; a + \epsilon)$ is a finite simplicial complex; so I_a^b has finite rank.

The proof for the Čech persistence module is the same. □

The above proposition implies that the persistence diagrams of $H(\check{\text{Cech}}(X))$ and $H(\mathbb{R}\text{ips}(X))$ are well-defined for precompact metric spaces. The persistence stability theorem 2.2 gives us the following result relating the Gromov–Hausdorff distance between two spaces to the bottleneck distance between the persistence diagrams of their Čech and Vietoris–Rips filtrations.

Theorem 5.2. *Let X, Y be precompact metric spaces. Then*

$$\begin{aligned} d_b(\text{dgm}(H(\check{\text{Cech}}(X))), \text{dgm}(H(\check{\text{Cech}}(Y)))) &\leq 2d_{\text{GH}}(X, Y), \\ d_b(\text{dgm}(H(\mathbb{R}\text{ips}(X))), \text{dgm}(H(\mathbb{R}\text{ips}(Y)))) &\leq 2d_{\text{GH}}(X, Y). \end{aligned}$$

Remark. The second inequality of Theorem 5.2 has already been proven in [5] is the particular case of finite metric spaces using a different approach based upon the embedding of finite metric spaces into l^∞ .

5.2 Dimension of homology groups of Rips and Čech filtrations on precompact metric spaces

Although the persistence modules $H(\check{\text{Cech}}(X))$ and $H(\text{Rips}(X))$ are q-tame when X is a precompact metric space, the homology groups $H(\check{\text{Cech}}(X, a))$ and $H(\text{Rips}(X, a))$ are not necessarily finite dimensional even when X is compact. In this section we discuss and illustrate a few properties about the dimensionality of these groups.

Homology groups of the Rips filtration

It is easy to construct an example of a compact metric space X such that for a given $a > 0$, $H_1(\text{Rips}(X, a))$ has an uncountable infinite dimension. For example consider the union X of two parallel segments in \mathbf{R}^2 defined by

$$X = \{(t, 0) \in \mathbf{R}^2 : t \in [0, 1]\} \cup \{(t, a) \in \mathbf{R}^2 : t \in [0, 1]\}$$

endowed with the restriction of the euclidean metric in \mathbf{R}^2 . Then for any $t \in [0, 1]$, $e_t = [(t, 0), (t, a)]$ is an edge of $\text{Rips}(X, a)$ but there is no triangle in $\text{Rips}(X, a)$ that contains e_t in its boundary. As a consequence, the cycles $\gamma_t = [(0, 0), (t, 0)] + e_t + [(t, a), (0, a)] - e_0$, $t \in (0, 1]$, are not homologous to 0 and are linearly independent in $H_1(\text{Rips}(X, a))$. Note that a is the only value for which the first homology group of the Vietoris–Rips complex $H_1(\text{Rips}(X, \cdot))$ is not finite. However in general the set of values a such that the first homology group is infinite dimensional can be arbitrarily large.

Proposition 5.3. *For any $\alpha, \beta \in \mathbf{R}$ such that $0 < \alpha \leq \beta$ and any integer k there exists a compact metric space X such that for any $a \in [\alpha, \beta]$, $H_k(\text{Rips}(X, a))$ has a non countable infinite dimension.*

Proof. The following example has been obtained with the help of J.-M. Droz who also proved that a similar example can be realised as a subset of \mathbf{R}^4 endowed with the Euclidean metric [8].

Without loss of generality, we can assume that $\alpha = 1$ and $\beta = 2$. Let us first consider the case $k = 1$. Consider the union X of the two non-parallel rectangles in \mathbf{R}^3 defined by

$$\begin{aligned} X &= R_1 \cup R_2 \\ &= \{(t, 0, z) \in \mathbf{R}^3 : t \in [0, 2], z \in [0, 1]\} \cup \{(t, 1 + \frac{1}{2}t, z) \in \mathbf{R}^3 : t \in [0, 2], z \in [0, 1]\} \subset \mathbf{R}^3 \end{aligned}$$

endowed with the restriction of the L^1 -norm in \mathbf{R}^3 (see Figure 1).

Since we are using the L^1 -norm, for $a \in [1, 2]$ and $z \in [0, 1]$, the point $(2(a - 1), 0, z) \in R_1$ is at distance a from R_2 and its unique closest point on R_2 is $(2(a - 1), a, z)$. As a consequence $e_z = [(2(a - 1), 0, z), (2(a - 1), a, z)]$ is an edge of $\text{Rips}(X, a)$ for all $z \in [0, 1]$ but there is no (non degenerate) triangle in $\text{Rips}(X, a)$ that contains e_z in its boundary. As a consequence, the cycles

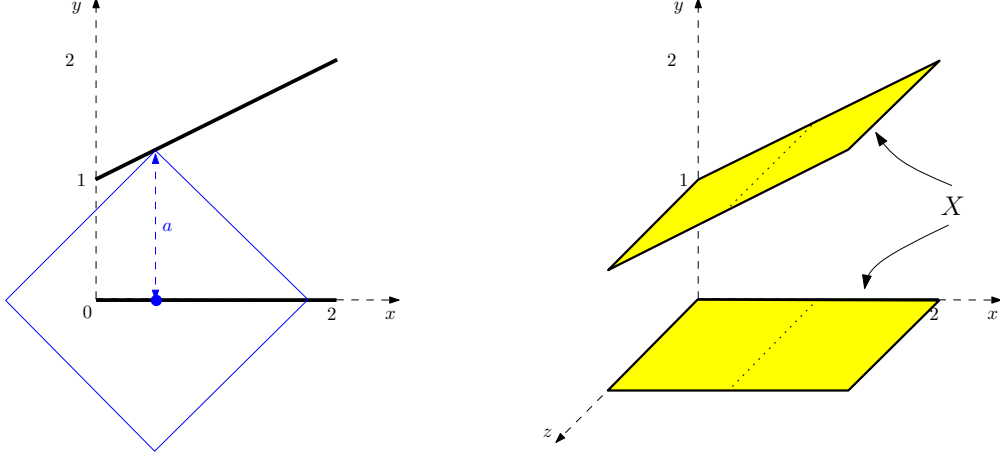


Figure 1: The union X of the above 2 rectangles endowed with the restriction of the L^1 metric in \mathbf{R}^3 provides an example of a compact metric space such that $\dim H_1(\text{Rips}(X, a)) = +\infty$ for any $a \in [1, 2]$.

$\gamma_z = [(2(a-1), 0, 0), (2(a-1), 0, z)] + e_z + [(2(a-1), a, z), (2(a-1), a, 0)] - e_0, z \in (0, 1]$ are not homologous to 0 and are linearly independent in $H_1(\text{Rips}(X, a))$.

To prove the lemma for $k > 1$, just consider the product of X with a $(k-1)$ -dimensional sphere of sufficiently large radius (to prevent the Rips construction from killing the $(k-1)$ -homology), and apply the Künneth formula [13, Theorem 3.16, p.219]. \square

The first homology groups of the Čech filtrations

As for Vietoris–Rips filtrations, the homology groups of the Čech filtration of a compact set might be infinite dimensional as shown by [2] (Appendix B), except for the first homology groups. The following result was originally obtained by Smale et al. [2, Theorem 8] with a different proof.

Proposition 5.4. *Let (X, d) be a precompact metric space, and let $a \geq 0$. Then, over any coefficient ring \mathbf{A} and any $a \in \mathbf{R}$, the 1-dimensional homology $H_1(\check{\text{Cech}}(X, a); \mathbf{A})$ is finitely generated over \mathbf{A} . In particular, over a field \mathbf{k} ,*

$$\dim_{\mathbf{k}}(H_1(\check{\text{Cech}}(X, a); \mathbf{k})) < +\infty.$$

Proof. The proof of the lemma follows from a sequence of elementary remarks.

1. Every 1-cycle in $\check{\text{Cech}}(X, a)$ is homologous to a 1-cycle whose edges have length at most a .

Proof: Any edge $[x, x']$ belonging to $\check{\text{Cech}}(X, a)$ has an a -centre; that is, a point y which satisfies $d(x, y) \leq a$ and $d(x', y) \leq a$. Since $d(y, y) = 0 \leq a$, the point y is also an a -centre for the triangle $[x, y, x']$ and the edges $[x, y]$, $[y, x']$. It follows that any 1-cycle

$$\gamma = \sum_i a_i [x_i, x'_i]$$

can be replaced by a homologous 1-cycle

$$\hat{\gamma} = \gamma + \partial \sum_i a_i [x_i, y_i, x'_i] = \sum_i a_i ([x_i, y_i] + [y_i, x'_i])$$

all of whose edges $[x_i, y_i]$, $[y_i, x'_i]$ have length at most a . □

2. There exists a finite set E_a of edges of length at most a , with the following property: for any edge $[x, y]$ of length at most a , there exists an edge $[x', y']$ in E_a such that $d(x, x') \leq a$ and $d(y, y') \leq a$.

Proof: Since X is totally bounded, so is $X \times X$ with the ℓ^∞ product metric

$$d((x, y), (x', y')) = \max(d(x, x'), d(y, y')).$$

Since $X \times X$ is totally bounded, so is its subspace

$$[X \times X]_a = \{(x, y) \in X \times X \mid d(x, y) \leq a\}.$$

Let $(x'_1, y'_1), \dots, (x'_N, y'_N)$ be an a -sample for $[X \times X]_a$. Then

$$E_a = \{[x'_1, y'_1], \dots, [x'_N, y'_N]\}$$

satisfies the required condition. □

3. Any 1-cycle can be written as a finite linear combination of cycles of the form

$$[x_1, x_2] + [x_2, x_3] + \dots + [x_{k-1}, x_k] + [x_k, x_1] \tag{*}$$

(k may vary).

Proof: This is standard, but we give the proof explicitly. Certainly any 1-cycle γ can be written as a finite linear combination of cycles (as above) and paths of the form

$$[x_1, x_2] + [x_2, x_3] + \dots + [x_{k-1}, x_k] + [x_k, x_{k+1}], \quad x_1 \neq x_{k+1},$$

(the trivial solution is to use paths of length 1 and no cycles). Consider the ‘free’ vertices in such a decomposition for γ : that is, vertices that occur as endpoints of the paths in the decomposition. We can eliminate the free vertices one by one as follows. Pick a free vertex and enumerate the paths which terminate there: P_1, P_2, \dots, P_m . Since $\partial\gamma = 0$, we must have $m \geq 2$. We can decrease m strictly by concatenating P_m with the appropriate multiple of P_{m-1} or its reverse. This creates a new, longer path (or cycle, if the endpoints coincide) in place of P_m , and rescales or annihilates P_{m-1} . Eventually $m = 0$ and the free vertex is eliminated. Finally, when there are no free vertices the decomposition involves only cycles, and we are done. □

4. Consider a cycle of the form

$$\gamma = [x_1, x_2] + [x_2, x_3] + \dots + [x_{k-1}, x_k] + [x_k, x_1]$$

whose edges have length at most a . Then γ is homologous in $\check{C}ech(X, a)$ to a cycle whose edges belong to E_a .

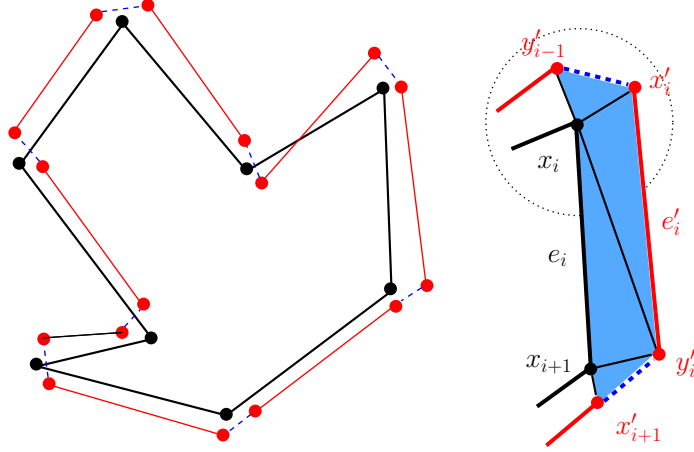


Figure 2:

Proof: Approximate each edge $[x_i, x_{i+1}]$ by an edge $[x'_i, y'_i] \in E_a$ according to remark 2 (interpreting x_{k+1} as x_1 , cyclically). We claim that

$$\hat{\gamma} = [x'_1, y'_1] + [y'_1, x'_2] + [x'_2, y'_2] + [y'_2, x'_3] + \cdots + [x'_k, y'_k] + [y'_k, x'_1]$$

is homologous to γ in $\check{\text{Cech}}(X, a)$. Indeed

$$\gamma - \hat{\gamma} = \partial \sum_{i=1}^k \left([x'_i, x_i, y'_i] + [y'_i, x_i, x_{i+1}] + [y'_i, x_{i+1}, x'_{i+1}] \right)$$

(see figure 2). To verify that the right-hand side of the equation belongs to $\check{\text{Cech}}(X, a)$, note that the triangles $[x'_i, x_i, y'_i]$, $[y'_i, x_i, x_{i+1}]$ and $[y'_i, x_{i+1}, x'_{i+1}]$ have a -centres x'_i , x_{i+1} and x_{i+1} respectively. \square

Combining remarks 1, 3 and 4, we see that in $\check{\text{Cech}}(X, a)$ any 1-cycle γ is homologous to a 1-cycle involving only edges in the finite set E_a . It follows that the homology $H_1(\check{\text{Cech}}(X, a); \mathbf{A})$ is finitely generated. \square

The open Vietoris–Rips filtration

The above examples provided to prove that the first homology groups of the Vietoris–Rips filtration of a compact set X might be infinite dimensional strongly rely on the fact that the definition of $\text{Rips}(X, a)$ involves a non-strict inequality, that is to say, a closed condition: $[x_0, x_1, \dots, x_k] \in \text{Rips}(X, a)$ if and only if $d_X(x_i, x_j) \leq a$, for all i, j .

It is thus natural to ask what happens if the strict inequality—an open condition—is used. Given a metric space (X, d_X) and a real number $a \in \mathbf{R}$, the open Vietoris–Rips complex is the simplicial complex $\text{Rips}(X, a^-)$ with vertex set X defined by the following condition:

$$[x_0, x_1, \dots, x_k] \in \text{Rips}(X, a^-) \iff d_X(x_i, x_j) < a, \text{ for all } i, j.$$

The open condition gives us the following mild regularity result.

Proposition 5.5. *For any precompact metric space X , any $a > 0$ and any integer k , the space $H_k(\text{Rips}(X, a^-))$ has a countable basis.*

Proof. Any homology class $H_k(\text{Rips}(X, a^-))$ is represented by some cycle c , a finite linear combination of simplices of diameter strictly less than a , and therefore bounded away from a . It follows that the class is in the image of $H_k(\text{Rips}(X, a - \frac{1}{n})) \rightarrow H_k(\text{Rips}(X, a^-))$ for some positive integer n . Since $H_k(\mathbb{R}\text{ips}(X))$ is q -tame, by Proposition 5.1, this image is finite dimensional. Now $H_k(\text{Rips}(X, a^-))$ is the union of these finite dimensional images for $n \rightarrow +\infty$, so it has a countable basis. \square

We still cannot guarantee finite dimensionality. However, Proposition 5.5 suggests that we must proceed discretely if we are to find a counterexample.

Proposition 5.6. *For any given $a > 0$ there exists a precompact metric space X such that $H_1(\text{Rips}(X, a^-))$ has infinite dimension.*

Proof. Without loss of generality we assume that $a = 1$. We will construct a bounded subset $X \subset \mathbf{R}^2$ whose open Vietoris–Rips complex $\text{Rips}(X, 1^-)$ has infinite-dimensional 1-dimensional homology. We will construct it as the union of two infinite sets L and R such that $\text{Rips}(X, 1^-)$ contains the complete graphs on L and R , and otherwise each vertex in L shares an edge with precisely one vertex in R , and vice versa. Following the same argument as the one used for the example before Proposition 5.3 we will deduce that $H_1(\text{Rips}(X, 1^-))$ is infinite dimensional.

We define X in terms of an auxiliary function $f(x) : [0, \infty) \rightarrow [0, \infty)$, which will be identified later. We suppose initially that f is continuous, increasing, and positive except at $f(0) = 0$. Specifically, let

$$L_x = (f(x), x), \quad R_x = (1 - f(x), x)$$

for $0 \leq x < f^{-1}(1/2)$. We define

$$X = \{L_x, R_x \mid x = \epsilon_n, n \gg 0\}$$

where (ϵ_n) is a decreasing positive sequence with limit 0.

Clearly $|L_x - R_x| < 1$ for all $x > 0$. We must arrange that $|L_x - R_y| \geq 1$, for x, y distinct elements of the sequence (ϵ_n) . Suppose $x > y$. Then:

$$\begin{aligned} |L_x - R_y|^2 &= (1 - f(x) - f(y))^2 + (x - y)^2 \\ &\geq (1 - 2f(x))^2 + (x - y)^2 \\ &\geq 1 - 4f(x) + (x - y)^2. \end{aligned}$$

Suppose we have chosen our sequence so that $x > y$ implies $x \geq 2y$; for instance, by setting $(\epsilon_n) = (2^{-n})$. Then

$$(x - y)^2 \geq \frac{1}{4}x^2.$$

Now we choose $f(x) = \frac{1}{16}x^2$. Then for $\sqrt{32} > x > y$ in the sequence (2^{-n}) we have

$$|L_x - R_y|^2 \geq 1 - 4f(x) + (x - y)^2 \geq 1 - \frac{1}{4}x^2 + \frac{1}{4}x^2 = 1$$

as required.

It follows that if we define

$$X = L \cup R = \{(2^{-2n-4}, 2^{-n}) \mid n \geq 1\} \cup \{(1 - 2^{-2n-4}, 2^{-n}) \mid n \geq 1\}$$

then $\text{Rips}(X, 1^-)$ contains the complete graphs on L and R , and otherwise each vertex in L shares an edge with precisely one vertex in R , and vice versa. As a consequence there is no non-degenerate triangle in $\text{Rips}(X, 1^-)$ that contains one of these edges connecting L to R . For each point $x_n = (2^{-2n-4}, 2^{-n}) \in L$, let $y_n \in R$ be the corresponding point in R , so the edge $e_n = [x_n, y_n]$ is in $\text{Rips}(X, 1^-)$. It follows that the cycles $\gamma_n = e_1 + [y_1, y_n] - e_n + [x_n, x_1]$, $n \geq 2$, are not homologous to 0 and are linearly independent. So $\dim H_1(\text{Rips}(X, 1^-)) = +\infty$. \square

The persistence diagram of $H_1(\mathbb{R}\text{ips}(X))$ when X is a path metric space

Recall that a metric space (X, d_X) is a path metric space if the distance between each pair of points is equal to the infimum of the lengths of the curves joining these two points.³ Every path metric space (X, d_X) satisfies the following property (see [12], Theorem 1.8): for any $x, x' \in X$ and any $\varepsilon > 0$ there exists a point $z \in X$ such that

$$\sup(d_X(x, z), d_X(x', z)) \leq \frac{1}{2}d_X(x, x') + \varepsilon.$$

Lemma 5.7. *Let (X, d_X) be a path metric space. For any $b > 0$, any 1-cycle in $\text{Rips}(X, b)$ is homologous to a 1-cycle in $\text{Rips}(X, \frac{2}{3}b)$.*

Proof. Any 1-cycle γ in $\text{Rips}(X, b)$ can be written as a finite sum $e_1 + \dots + e_n$, of edges $e_i = [x_i, x'_i]$ of length at most b . For any i , there exists $z_i \in X$ such that:

$$\begin{aligned} d_X(x_i, z_i) &\leq \frac{1}{2}d_X(x_i, x'_i) + \varepsilon \leq \frac{2}{3}b \\ d_X(x'_i, z_i) &\leq \frac{1}{2}d_X(x_i, x'_i) + \varepsilon \leq \frac{2}{3}b \end{aligned}$$

As a consequence the triangle $[x_i, x'_i, z_i]$ is contained in $\text{Rips}(X, b)$ and γ is homologous in $\text{Rips}(X, b)$ to

$$\gamma' = \sum_{i=1}^n [x_i, z_i] + [z_i, x'_i]$$

which is a 1-cycle in $\text{Rips}(X, \frac{2}{3}b)$. \square

Corollary 5.8. *Let (X, d_X) be a path metric space. Then the map $H_1(\text{Rips}(X, a)) \rightarrow H_1(\text{Rips}(X, b))$ is surjective, for all $0 < a < b$.*

Proof. Iterate the surjectivity of $H_1(\text{Rips}(X, \frac{2}{3}b)) \rightarrow H_1(\text{Rips}(X, b))$. Eventually $(\frac{2}{3})^n b \leq a$. \square

From the corollary we immediately deduce (using the rectangle-measures and quiver calculus of [6]) the following result on the structure of the persistence diagram of $H_1(\mathbb{R}\text{ips}(X))$.

³See [12] chap.1, def 1.2 for a definition of the length of a curve in a metric space.

Proposition 5.9. *Let (X, d_X) be a precompact path metric space. Then the persistence diagram of $H_1(\mathbb{Rips}(X))$ is contained in the vertical line $\{0\} \times [0, +\infty)$.*

Proof. It is enough to show that $\mu(R) = 0$ for any rectangle which does not meet the line. Write $R = [a, b] \times [c, d]$, where $-\infty \leq a < b < c < d \leq +\infty$. Either $b < 0$, in which case

$$\mu(R) = \langle \circ_a \text{---} \bullet_b \text{---} \bullet_c \text{---} \circ_d \rangle \leq \langle \bullet_b \rangle = \dim H_1(\mathbb{Rips}(X, b)) = 0$$

because there are no edges when $b < 0$. Or else $0 < a$, in which case

$$\mu(R) = \langle \circ_a \text{---} \bullet_b \text{---} \bullet_c \text{---} \circ_d \rangle \leq \langle \circ_a \text{---} \bullet_b \rangle = 0$$

because $H_1(\mathbb{Rips}(X, a)) \rightarrow H_1(\mathbb{Rips}(X, b))$ is surjective. □

Remark. Note that Proposition 5.9 relies on the fact that in a length space any 1-dimensional simplex can be “subdivided” into a sum of smaller simplices, i.e. is homologous to a sum of simplices of strictly smaller diameter. This property does no longer hold for higher dimensional simplices. For example if (X, d_X) is a circle of length 1, then any triple of points x, y, z such that $d_X(x, y) = d_X(y, z) = d_X(z, x) = \frac{1}{3}$ spans a triangle in $\mathbb{Rips}(X, \frac{1}{3})$ that is not homologous to any finite sum of triangles of diameter strictly less than $\frac{1}{3}$. However Proposition 5.9 can be directly extended to the case of δ -hyperbolic metric spaces (see [11] Chap.2 for a definition) where one can then prove that the persistence diagram of $H_2(\mathbb{Rips}(X))$ is contained in a vertical band of the form $O(\delta) \times [0, +\infty)$. The proof of this result is beyond the scope of this paper, but not too difficult; so it is left to the reader.

Acknowledgements

The authors thank Steve Smale for fruitful discussions that motivated the results Section 5.2. They also thank J.-M. Droz for suggesting the idea of the proof of Proposition 5.3. They acknowledge the support of the Digiteo Project C3TTA (including the Digiteo chair held by the second author); the European project CG-Learning EC contract No. 255827; the ANR project GIGA (ANR-09-BLAN-0331-01); and the DARPA project Sensor Topology and Minimal Planning (SToMP): HR0011-07-1-0002.

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