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On the extrinsic nature of triangulations

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Abstract The extrinsic nature of the complexity of triangulations and more generally approximations in higher codimension is exhibited.

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Introduction For practical applications, such as in animations, it is often necessary to find triangulations of surfaces, which are both accurate and easy to handle. The accuracy has been formalized by the concept of Hausdorff distance between two sets in a metric space:

$$d_H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\}.$$

The ease of use will be simply be taken to mean involving a small number of vertices. Fejes Tóth [1] (in three dimensions) and Schneider [3] have shown that ratio of number of vertices used in an optimal triangulation T_m of a convex manifold M in dimension $n-1$ and the Hausdorff distance $d_H(M,T_m)$ between the manifold and the triangulation is given by

$$\lim_{m\to\infty} m d_H(M,T_m)^{(n-1)/2} = 2^{(1-n)/2}\kappa_{n-1}^{-1}\vartheta_{n-1}\int_M \sqrt{K}\mathrm{d}M,$$

where $\kappa_n = \pi^{n/2}/\Gamma(1+n/2)$ is the volume of the n-dimensional unit ball, ϑ_n the covering density of the ball in n-dimensional space and K the Gaussian curvature. This is often referred to as the complexity. This formula is intrinsic in nature, as the Gaussian curvature is. We explicitly have that (chapter 7 of [5]):

$$K = \frac{1}{2^{n/2}n!}\sum_{i_1,\dots,i_n}\sum_{j_1,\dots,j_n} R_{i_1i_2j_1j_2}\dots R_{i_{n-1}i_nj_{n-1}j_n}\frac{\epsilon^{i_1\dots i_n}}{\sqrt{\det(g_{ij})}}\frac{\epsilon^{j_1\dots j_n}}{\sqrt{\det(g_{ij})}},$$

where R_{ijkl} denotes the Riemann tensor, g_{ij} the metric and $\epsilon^{i_1\cdots i_n}$ the Levi-Civita symbol. The intrinsic nature of the complexity is particular to low codimension. This is heuristically clear because the rigidity of a manifold disappears in the codimension of the embedding is sufficiently high, as was noted by Nash in his seminal paper [2]. Nash in fact proved that a compact n-manifold with a C^k positive metric has a C^k isometric embedding in any small volume of Euclidean $(n/2)(3n+11)$-space, provided $3 < k \leq \infty$. So roughly speaking, one can squash a manifold in a small volume without affecting the curvatures but this would lead to, for lack of a better word, wrinkles. This is opposite to the results for surfaces in three dimensional Euclidean space, where strong rigidity results are available, see [4, 6].

Result Below, we exhibit a family of isometric embeddings of the flat torus of such that

$$\lim_{m\to\infty} d_H(E_k,T_m)m = c_{E_k},$$

where E_k indicates a member of the family of embeddings of $S^1\times S^1$ parameterized $k\in\mathbb{Z}_{\geq 1}$, T_m is an optimal triangulation with m vertices on E_k, d_H indicates the Hausdorff distance and c_{E_k} is a real number depending on k, such that

$$\lim_{k\to\infty} c_{E_k} = \infty.$$

Before doing so, we note that although in general approximating surfaces by piecewise quadratic pieces, or more general pieces of a algebraic surface of higher order, is not so well understood the dependence on the exact isometric embedding can be shown much more easily as for piecewise linear approximation. Here we study the piecewise quadratic approximations of the flat torus where the embedded as

$$E_0 = \{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 | x_1^2+x_2^2=1, x_3^2+x_4^2=1\}$$
$$E_1 = \{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 | (x_1/c)^2+(x_2/c)^4=1, (x_3/c)^2+(x_4/c)^4=1\},$$

where c is a constant such that the surface area of both embeddings are the same. We depict $\{(x_1,x_2)\in\mathbb{R}^2|(x_1/c)^2+(x_2/c)^4=1\}$ in figure 1.

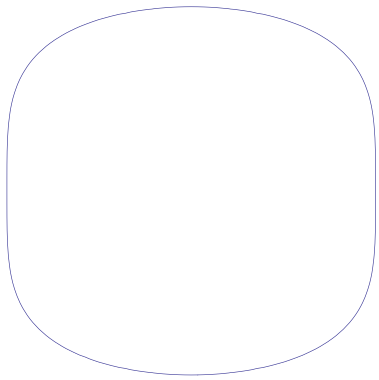

Figure 1: The set $\{(x_1, x_2) \in \mathbb{R}^2 | (x_1/c)^2 + (x_2/c)^4 = 1\}$.

What we in fact use can be easily seen in three dimensions. If we take a sheet of paper and fold and glue it into a cylinder $\{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\} \times [a, b]$, with $a < b$, we can deform the cylinder by deforming $\{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$ changing the algebraic properties while the angles on the sheet of paper remain the same and thus the intrinsic geometry does not change. The first of these two embeddings is clearly an algebraic surface of order 2 so one pieces of a quadratic surface, namely the surface itself, suffices to approximate the surface that is

$$d_H(E_0, Q_1) = 0,$$

where Q_m is a piecewise quadratic surface with m patches which approximates the surface optimally. The second embedding of the flat torus is a quartic surface so there is no finite m such that

$$d_H(E_1, Q_m) = 0.$$

We could try and show the effect introduced above for the following family of embeddings of the flat torus which for the first two coordinates reads

$$\{(1 + \cos(2k\theta)/20)(\cos(\theta), \sin(\theta))/r_k | \theta \in [0, 2\pi]\},$$

where we have set the length of the topological circle to 2π by taking

$$r_k = \int_0^{2\pi} \frac{1}{40\sqrt{2}\pi} \sqrt{801 + 4k^2 + 80\cos(2k\theta) + (1 - 4k^2)\cos(4k\theta)} \mathrm{d}\theta.$$

And likewise for the last two coordinates. However since the extrinsic curvature of a member of this family is strongly dependent on the coordinate θ, which makes estimates very difficult and thus the example less clear, we shall not use this family of embeddings in $\mathbb{R}^4$ but focus on embeddings in $\mathbb{R}^8$. We shall study the family of embeddings of the flat torus E_k parameterized by $k \in \mathbb{Z}_{\geq 1}$

$$\{(\cos(\theta), \sin(\theta), \cos(k\theta)/k, \sin(k\theta)/k, \cos(\varphi), \sin(\varphi), \cos(k\varphi)/k, \sin(k\varphi)/k) | \theta \in [0, 2\pi], \varphi \in [0, 2\pi]\}.$$

Firstly we note that since the surface contains no straight lines, for each k, we have that the edge length of each edge in a triangulation T_m tends to zero as $d_H(T_m, E_k)$ tends to zero. This implies that we may use local approximations and the tangent plane of the surface in the neighbourhood of a triangle is asymptotically well defined. Secondly we we may employ the natural group action of $(\mathrm{SO}(2))^4$ to shift a given point on the torus to the origin. This means that we can approximate the surface by

$$(1 - \theta^2/2, \theta, 1 - \theta^2/(2k), \theta, 1 - \varphi^2/2, \varphi, 1 - \varphi^2/(2k), \varphi)$$

and thus through a translation by

$$\Sigma_k(\theta,\varphi) \simeq (-\theta^2/2, \theta, -\theta^2/(2k), \theta, -\varphi^2/2, \varphi, -\varphi^2/(2k), \varphi).$$

Furthermore we may assume that the vertices of a triangle are $\Sigma_k(0,0)=0$, $\Sigma_k(\theta_1,\varphi_1)$, $\Sigma_k(\theta_2,\varphi_2)$. We shall employ similar techniques to the ones employed by Fejes Tóth [1] to find a a lower bound for

$$\lim_{m\to\infty} d_H(T_m, E_k)m.$$

Roughly speaking Fejes Tóth considered a family of triangles whose one-sided Hausdorff distance to the surface had a given value and then consecutively determined the which member of the family had the greatest area. Because the surface is approximated by triangles one can infer a lower bound on the number of triangles in the triangulation based on these considerations. Because we are only interested in lower bounds it suffices to fixate some lower bound on the Hausdorff distance and some upper bound on the area of the triangles satisfying this bound. To determine the Hausdorff distance we firstly determine for a given point $p = \Sigma_k(\theta_1,\varphi_1)\lambda_1 + \Sigma_k(\theta_2,\varphi_2)\lambda_2$, where $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0, 1-\lambda_1]$, the point on the surface $\Sigma_k(\theta_c,\varphi_c)$ which is closest. We optimize in the usual manner, that is we impose

$$\partial_\theta |p - \Sigma_k(\theta,\varphi)|^2 = 0$$
$$\partial_\varphi |p - \Sigma_k(\theta,\varphi)|^2 = 0.$$

It is not difficult to verify that $\theta_c \simeq \theta_1\lambda_1 + \theta_2\lambda_2$ and $\varphi_c \simeq \varphi_1\lambda_1 + \varphi_2\lambda_2$, where $\simeq$ denotes equality up to leading order. So that the distance between a point on the triangle and the surface is given by

$$\begin{aligned}
&\|\Sigma_k(\theta_1,\varphi_1)\lambda_1 + \Sigma_k(\theta_2,\varphi_2)\lambda_2 - \Sigma_k(\theta_1\lambda_1+\theta_2\lambda_2, \varphi_1\lambda_1+\varphi_2\lambda_2)\| \\
&\quad = \left\| \Big((\theta_1\lambda_1+\theta_2\lambda_2)^2/2 - \frac{\theta_1^2}{2}\lambda_1 - \frac{\theta_2^2}{2}\lambda_2, 0, k(\theta_1\lambda_1+\theta_2\lambda_2)^2/2 - k\frac{\theta_1^2}{2}\lambda_1 - k\frac{\theta_2^2}{2}\lambda_2, 0, \right. \\
&\qquad \left. (\varphi_1\lambda_1+\varphi_2\lambda_2)^2/2 - \frac{\varphi_1^2}{2}\lambda_1 - \frac{\varphi_2^2}{2}\lambda_2, 0, k(\varphi_1\lambda_1+\varphi_2\lambda_2)^2/2 - k\frac{\varphi_1^2}{2}\lambda_1 - k\frac{\varphi_2^2}{2}\lambda_2 \Big) \right\|.
\end{aligned}$$

For $\lambda_1 = 1/2$, $\lambda_2 = 0$ and $\lambda_1 = 0$, $\lambda_2 = 1/2$ this yields $\sqrt{1+k^2}\sqrt{\theta_1^4+\varphi_1^4}/8$ and $\sqrt{1+k^2}\sqrt{\theta_2^4+\varphi_2^4}/8$ respectively. So that

$$d_H \geq \eta = \max\left\{ \sqrt{1+k^2}\sqrt{\theta_1^4+\varphi_1^4}/8,\ \sqrt{1+k^2}\sqrt{\theta_2^4+\varphi_2^4}/8 \right\}.$$

The area of the triangle on the other hand is approximately equal to

$$|\varphi_1\theta_2 - \theta_1\varphi_2|/2.$$

It is clear the area of a triangle on which a given η is attained is bounded from above by $42\eta/\sqrt{1+k^2}$ because $8\eta/\sqrt{1+k^2} \geq \theta_1, \theta_2, \varphi_1, \varphi_2$. Since furthermore the number of triangles $\tilde{m}$ in a triangulation is bounded below by

$$\tilde{m} \gtrsim \frac{\text{Area}(E_k)}{\text{Area}(\triangle)},$$

where E_k denotes the embedding of the surface and $\triangle$ denotes the biggest triangle in the triangulation. These considerations give us

$$d_H m \gtrsim \eta m \gtrsim \eta \frac{\text{Area}(E_k)}{\text{Area}(\triangle)} \gtrsim \eta \frac{(4\pi)^2}{42\eta/\sqrt{1+k^2}} = \frac{(4\pi)^2}{42}\sqrt{1+k^2}$$

So that

$$\lim_{m\to\infty} d_H(T_m, E_k)m \geq \frac{(4\pi)^2}{42}\sqrt{1+k^2}.$$

The result may be summarized in the following lemma

Lemma 1 *Let M a surface, then there is generally no function $f(g)$ which depends only on the metric and all its derivatives and a constant $\tilde{c}$ such that*

$$\lim_{m\to\infty} d_H(T_m, E(M))m \leq \tilde{c} \int_M f(g)\mathrm{d}A,$$

where $E(M)$ denotes the embedding of the manifold in Euclidean space.

In this lemma we could have absorbed $\tilde{c}$ in $f(g)$, we have however chosen this form to mimic the traditional form of the result of Fejes Tóth [1] and Schneider [3]. The generalization of the above lemma is trivial.

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