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Computational Geometric Learning

A conceptual take on invariants of low-dimensional manifolds found by integration

M.H.M.J. Wintraecken G. Vegter

CGL Technical Report No.: CGL-TR-43

Part of deliverable: WP-III/D3.2
Site: RUG
Month: 24

Project co-funded by the European Commission within FP7 (2010–2013)
under contract nr. IST-25582

Abstract An elementary proof of the fact that the Euler characteristic is the only topological invariant of a surface that can be found by integration (using Gauss-Bonnet) is given. A similar method is also applied to three dimensional manifolds.

Keywords: characteristic classes and numbers (57R20), low dimensional topology (57M99), Random fields; image analysis (62H35), Image analysis (62M40)

Note: Submitted to 'Topology and its Applications'.

Introduction Recently a lot of effort has been put into the the study of topological properties of Gaussian random fields [2, 6, 15], using among others Euler integration [4]. This is especially important in the context of cosmology, because these random fields are believed to describe the density fluctuations in the early universe [3] or are at least a good approximation of these fluctuations. The topological aspect has gained popularity over the last few years as there is a great deal of numerical evidence of the fact that the topology of the density field is very sensitive to small deviations from Gaussianity. A nice closed formula for the expectation value of the Euler characteristic of the level sets of these fields has been found [3, 8]. This result relies on the Gauss-Bonnet theorem which relates the integral of some intrinsic quantity whose origins lie in the field of differential geometry, namely the Gaussian curvature, to some topological invariant, the Euler characteristic. A result that can be generalized to higher dimensional manifolds the Gauss-Bonnet theorem can be generalized, using the theory of characteristic classes. For a very elegant exposition we refer to Milnor and Stasheff [12] or alternatively Spivak [13]. The generalization only goes so far, in fact Abrahamov [1] proved that the invariants thus produced (the so-called Chern numbers) are unique, up to some equivalence. See Gilkey [7] for a modern (and more extensive) treatment. This result implies that no expressions can be found for all other interesting topological invariants for Gaussian fields in three dimensions, such as Betti numbers, associated to the field, using a similar straightforward integration technique. Both the proof by Abrahamov and the modern treatment by Gilkey involve a significant amount of analysis and machinery. Below we provide a proof of a similar statement for two and three dimensional manifolds, which does not need to call in an elaborate set of calculations.

The formulation of the main result will be along the lines of the following question proposed by I.M. Singer: 'Suppose that f is a scalar valued invariant of the metric such that $t = \int f \mathrm{dvol}$ is independent of the metric. Then is there some universal constant c so that $t = c\chi(M)$?' This question has reportedly ([7]) been answered in the affirmative by E. Miller.
Our proof relies heavily on the classification of two dimensional closed surfaces and on Heegaard splitting. A discussion of the classification can be found in [9] or [10] , for the latter we refer to [5] or [14]. We complete our discussion by some remarks on generalizations.

The two dimensional case

Theorem 1 *Let M be an orientable[1] two dimensional real compact manifold and f a function on M which is completely determined by the metric and its derivatives, that is locally f can be written as $f(g, \partial g, \dots)$ with g the metric, such that*

$$\int_M f \,\mathrm{d}vol,$$

where $\mathrm{d}vol$ indicates the volume form, is a topological invariant t. Then there exists a real number c such that $t = c\,\chi(M)$, where χ denotes the Euler characteristic.

Proof First we note that the space of Riemannian metrics on a manifold is connected. This is obvious because if g and $\tilde{g}$ are metrics then so is $\lambda g + (1-\lambda)\tilde{g}$ for all $\lambda \in [0, 1]$. This means that we can assume without loss of generality that M is isometrically embedded in $\mathbb{R}^3$. Because we can choose $\tilde{g}$ to be the standard metric of M. Now let f be a function as described in the theorem, such that

$$\int_M f \,\mathrm{dvol} = t$$

is a topological invariant. Suppose that for the two sphere $\mathbb{S}^2$ we have

$$\int_{\mathbb{S}^2} f \,\mathrm{dvol} = 2c,$$

[1] Clearly the integral over a non-orientable manifold does not make sense.

where c is some constant. From this we can conclude that for the sphere $t = c\chi(M)$.

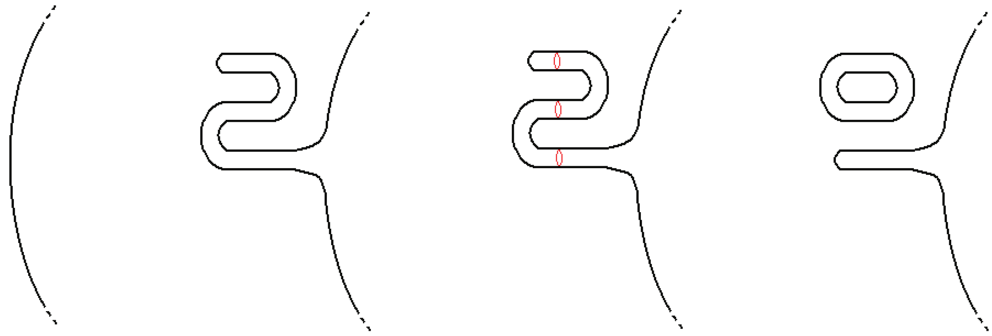

Figure 1: From left to right we have sketched: a part of the sphere, the same part of the sphere after deformation, the deformed surface with cutting lines indicated and the reassembled surfaces.

We can now deform the two-sphere as follows. A small region is pushed outwards and bent -in a sufficiently smooth manner- such that this region contains three equally spaced parallel cylinders pieces all of the same radius. We can now cut in the cylindrical part along the plane orthogonal to the cylinder and reassemble the parts so that we recover a topological sphere but also get a torus. The integral is not altered because integral are additive. The procedure is illustrated in figure 1. Because the integral is clearly additive for unions this implies that

$$\int_{C_1} f \mathrm{dvol} = 0,$$

where C_1 is a surface of genus 1. Generally we shall denote a surface of genus g by C_g.

The rest of the proof is inductive in nature. We begin with a topological genus-g torus and two spheres. We deform these surfaces so that the spheres contain a piece of a cylinder, both of the same radius, and the n-torus such that it contains two pieces of the cylinder, again of the same radius, so that if these pieces are deleted one of the remaining surfaces is itself a topological cylinder. We again cut the cylindrical pieces in half and reassemble the part so that we have a genus-$g-1$ torus and a sphere. As sketched in figure 2.

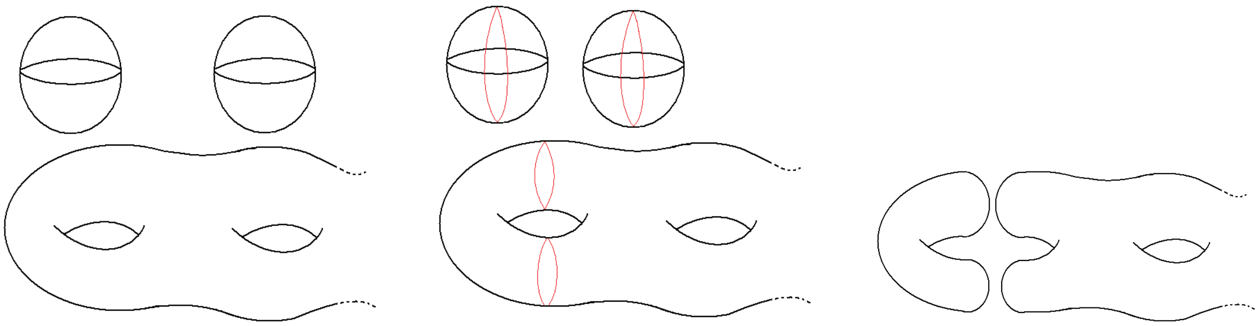

Figure 2: From left to right we have sketched: An n-torus and two spheres, the same surfaces with the lines along which we cut and the reassembled surfaces. In this figure we do not put emphasis on the deformation.

We can now conclude that

$$\int_{C_g} f \mathrm{dvol} + 2\int_{\mathbb{S}^2} f \mathrm{dvol} = \int_{C_{g-1}} f \mathrm{dvol} + \int_{\mathbb{S}^2} f \mathrm{dvol}$$

and thus by induction that

$$\int_{C_g} f \mathrm{dvol} = c(2-2g) = c\,\chi(C_g).$$

By the classification of all 2-manifolds we have proven the theorem for all two dimensional real manifolds embedded in $\mathbb{R}^3$. □

Three dimensions We will now focus on the three dimensional case. The intuition for the following proof is much strengthened by the remark that a morse function h on some manifold M can always be interpreted as height function. This can be easily seen as follows: Let M be isometrically embedded in $\mathbb{R}^n$, possibly using the Nash embedding theorem. Then we can add the value of the Morse function as another coordinate to a point $p \in M \subset \mathbb{R}^n$, so that the manifold M is embedded in $\mathbb{R}^{n+1}$ and the last coordinate is the height.

Theorem 2 *Let M be a three-dimensional real manifold and f a function which is completely determined by the metric and its derivatives such that*

$$\int_M f\, \mathrm{d}vol,$$

is a topological invariant t. Then we have $t = 0$.

Proof The first step in our proof will consist of showing that if $M = C_g \times \mathbb{S}^1$ we have that

$$\int_M f\, \mathrm{dvol} = 0.$$

To show this we shall consider any manifold N, that admits a Heegaard splitting of genus g. This means that the manifold N can be represented as the attachment of two three-dimensional manifolds, which are both homeomorphic to a three-dimensional ball with g handles, with respect to a diffeomorphism of their boundaries. We further have that there exists a Morse function h on N with one minimum and one maximum and all critical points of index 1, 2 correspond to the critical values c_1 and c_2 respectively with $c_1 < c_2$, see [5]. This has been schematically represented in the leftmost picture in figure 3. [2]

We now define for every surface C_g of genus g, some metric induced by an embedding in $\mathbb{R}^3$, exhibiting $\mathbb{Z}_2$ symmetry. We shall refer to this Riemannian manifold as the standard surface of genus g. In the following we view N as embedded in $\mathbb{R}^k$. Let f be as in theorem 1 such that

$$\int_N f\, \mathrm{dvol},$$

is a topological invariant t. For some sufficiently small $[a_1, b_1] \subset \mathbb{R}$, with $c_1 < a_1 < \alpha_1 < \beta_1 < b_1 < c_2$, we smoothly and isotopically deform $h^{-1}([a_1, b_1]) \cap M \sim C_g \times [a_1, b_1]$, so that $h^{-1}([\alpha_1, \beta_1]) \cap M$ becomes isometric to the standard $C_g \times [\alpha_1, \beta_1] \subset \mathbb{R}^4 \subset \mathbb{R}^k$ given by the standard C_g and the ordinary Cartesian product. This standard form is referred to as straight. We shall now deform this part of the manifold so that it consists of a straight piece and two pieces which are straight at the beginning and the end but

[2] Note that conversely a Heegaard splitting also gives a Morse function in a natural manner. Namely we start with Morse functions on both g-handled balls, by simply taking a Morse function on the standard g-handled ball and pulling back via the diffeomorphisms to the g-handled balls in question. Now theorem 1.4 and lemma 3.7 of [11], give a differentiable structure on the union with a smooth structure compatible with the given differentiable structure on the different parts, moreover such that the Morse functions on both parts piece together to a smooth function.

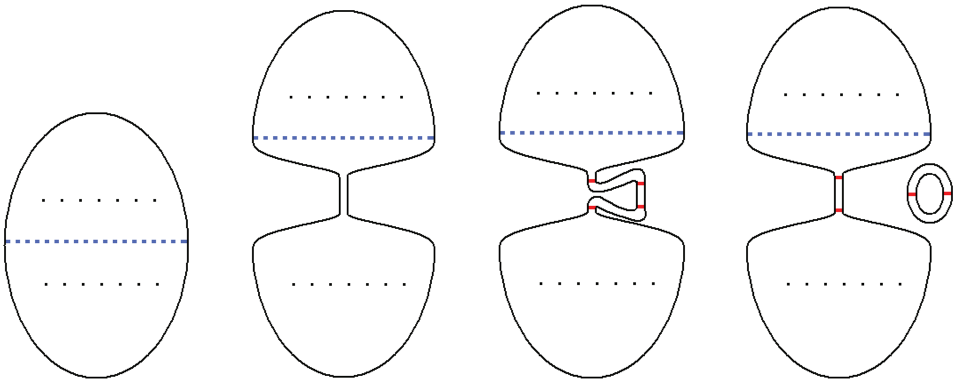

Figure 3: From left to right we have sketched: A manifold admitting a Heegaard splitting; the critical points of the Morse function are indicated as dots and the attachment by a blue dashed line, the same manifold with a small part of it brought to a standard $C_g \times [-\delta, \delta]$ metric, the deformed surface with cutting lines (red) indicated and the reassembled surfaces.

are bent in the middle so that if we cut along the the boundaries of the pieces and reassemble we recover the original manifold and $C_g \times \mathbb{S}^1$. The procedure is sketched in figure 3. From this we conclude that

$$\int_N f \,\mathrm{dvol} = \int_N f \,\mathrm{dvol} + \int_{C_g \times \mathbb{S}^1} f \,\mathrm{dvol},$$

where we again used local isotopy and the additivity of integration. Therefore,

$$\int_{C_g \times \mathbb{S}^1} f \,\mathrm{dvol} = 0.$$

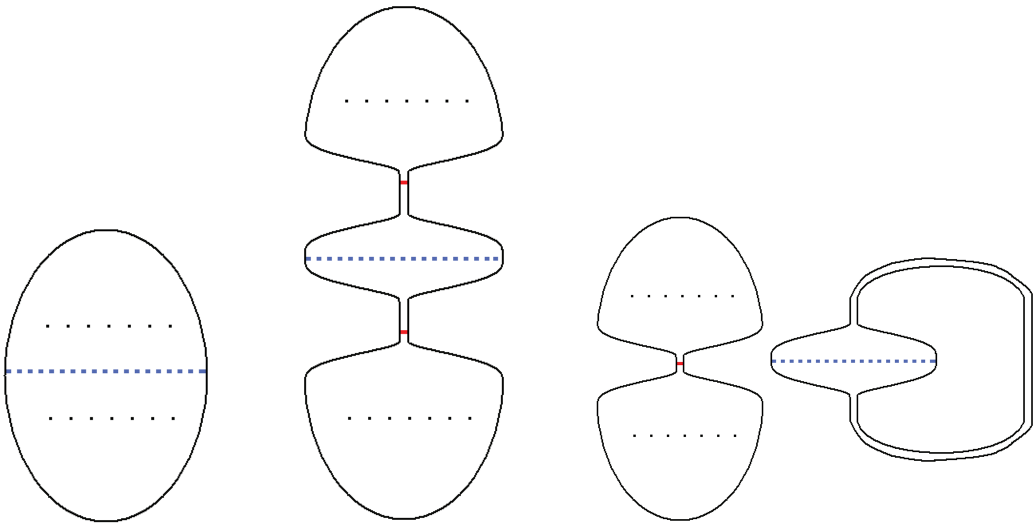

Figure 4: A Heegaard splitting, then the same manifold with two small parts brought to a standard metric both on another side of the 'attachment line', cutting lines (red) are also indicated, and finally the reassembled surface (two connected components).

The next part of the proof relies on the fact that the sphere ($\mathbb{S}^3$) allows a Heegaard splitting of every genus g, see [5]. Let M be a manifold which allows a Heegaard splitting of genus g. We now deform two pieces of the manifold into parts isometric to $C_g \times [\alpha_1, \beta_1]$ and $C_g \times [\alpha_2, \beta_2]$, with $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$, so that for all $p_1 \in (\alpha_1, \beta_1)$ and $p_2 \in (\alpha_2, \beta_2)$ both $h^{-1}((-\infty, p_1)) \cap M$ and $h^{-1}(p_2, \infty) \cap M$ are topological spheres with g handles whose boundary is isometric to the standard genus g surface, as discussed above. We can now smoothly deform $h^{-1}((p_1, q_1)) \cap M$ and $h^{-1}((q_2, p_2)) \cap M$, with $p_1 < q_1 < \beta_1$ and $\alpha_2 < q_2 < p_2$ (see figure 5), such that if we cut along the p_i and q_i lines and reassemble (possibly using $\mathbb{Z}_2$ symmetry) we recover two topological manifolds, with given topology. One of the manifolds we thus construct is a manifold admitting a Heegaard splitting of genus g. The attachment diffeomorphism, of the latter, on the boundary of the sphere with g handles is the identity. This manifold shall be denoted by $M_g^{S(3\mathrm{D})}$. The other manifold is a topological $C_g \times \mathbb{S}^1$-manifold. The entire procedure is sketched in figure 4.

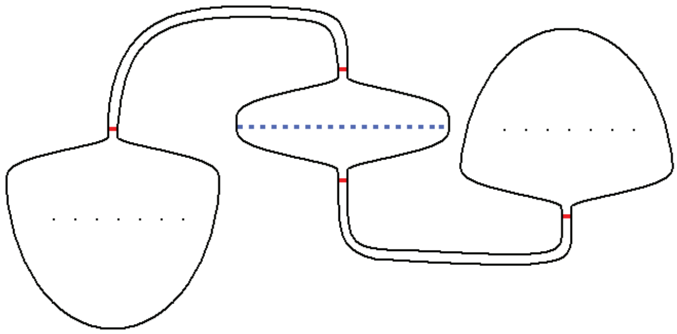

Figure 5: A sketch of the deformed manifold with the cutting lines (red) and the 'attachment line' (blue).

This means that by deforming, cutting and pasting a manifold M, which allows a Heegaard splitting of genus g, we find the following equalities

$$\int_M f\,\mathrm{dvol} = \int_{M_g^{S(3\mathrm{D})}} f\,\mathrm{dvol} + \int_{C_g \times \mathbb{S}^1} f\,\mathrm{dvol} = \int_{M_g^{S(3\mathrm{D})}} f\,\mathrm{dvol} + 0,$$

where f is as defined in the theorem. If we now use that the sphere ($\mathbb{S}^3$) allows a Heegaard splitting of every genus g we find that

$$\int_M f\,\mathrm{dvol} = \int_{M_g^{S(3\mathrm{D})}} f\,\mathrm{dvol} = \int_{\mathbb{S}^3} f\,\mathrm{dvol}. \tag{1}$$

Following this observation, we are able to use the result of the first part of the proof,

$$\int_{C_g \times \mathbb{S}^1} f\,\mathrm{dvol} = 0.$$

This immediately translates into

$$\int_{\mathbb{S}^2 \times \mathbb{S}^1} f\,\mathrm{dvol} = 0.$$

We notice that both $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$ allow a Heegaard splitting of genus 1, so that

$$\int_{\mathbb{S}^3} f\,\mathrm{dvol} = \int_{M_1^{S(3\mathrm{D})}} f\,\mathrm{dvol} = \int_{\mathbb{S}^2 \times \mathbb{S}^1} f\,\mathrm{dvol} = 0. \tag{2}$$

Combining equations (1) and (2) yields

$$\int_M f\,\mathrm{dvol} = 0,$$

for any manifold M and $f = f(g, \partial g, \ldots)$ a function determined by the metric and all its derivatives. $\square$

Discussion One can wonder about generalizations of the methods stated above to manifolds of general dimension. Some of these generalizations are immediately obvious, for example the procedure sketched in figure 3 can be used in any dimension so see that for f and t as in the theorem

$$\int_{M^{d-1}\times\mathbb{S}^1} f \,\mathrm{dvol} = t$$

implies that $t = 0$, where M^{d-1} is any manifold of dimension $d-1$ occurring as level set. However a full classification of all integrals yielding a topological invariant does not seem feasible because there is no easy classification of manifolds of dimension $d-1$ for $d > 3$ (and none for $d > 4$), occurring as the level sets of a Morse function on a manifold of dimension d.

Acknowledgements The authors are greatly indebted to Erik van den Ban for discussion and suggestions. We further acknowledge Rien van de Weijgaert and Ramsay Dyer for discussion and Arthemy Kiselev for discussion and use of his knowledge of Russian.

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