# On the Asymptotic Approximation of Manifolds embedded in Euclidean Space

## How burdensome is it to make Shrek look good?

16th of December 2011

# Motivation

Want:

- Accurate approximation
- Small number of points

Get by:

- More points if greatly detailed
- Few points if uncomplicated

Formalize by:

- Gaussian curvature

# Outline

Introduction

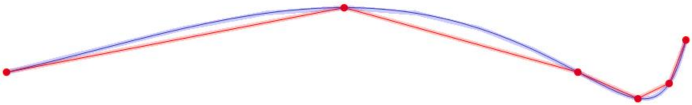
Approximating convex surfaces

Non-convex surfaces

# Introduction

# The Hausdorff distance

Hausdorff distance between $M_1, M_2 \subset \mathbb{R}^3$ is the smallest $\epsilon$ such that $M_1$ lies in an $\epsilon$-neighbourhood of $M_2$ and vice versa.



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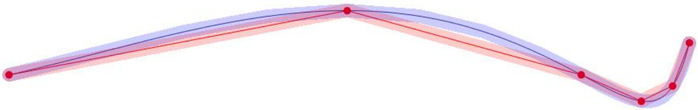

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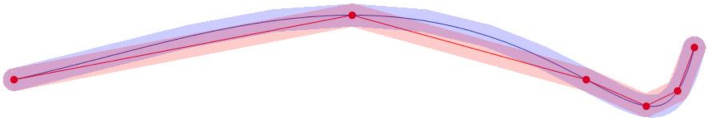

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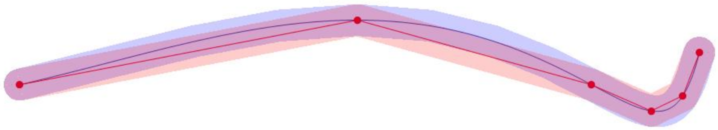
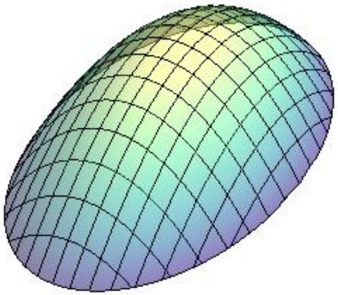
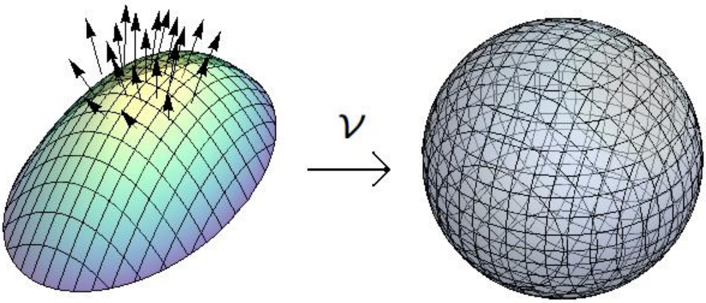


# Gaussian curvature

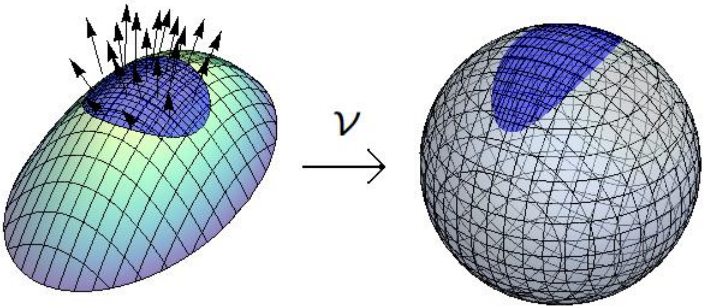
Surface $M \subset \mathbb{R}^3$



# Normal of surface: map from $M$ to $S^2 \subset \mathbb{R}^3$ (the Gauss map)



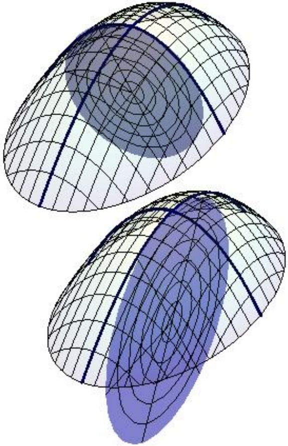
Patch $A$ on the surface and its image under $\nu$.




Gaussian curvature is

$$\lim_{A \to p} \frac{\text{oriented area } \nu(A)}{\text{oriented area } A} = K(p)$$

# Principal directions



Gaussian curvature is also

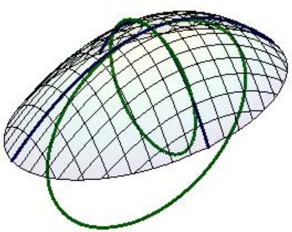
$$K = k_1 \cdot k_2,$$

where $k_1$, $k_2$ are principal curvatures

$$k_i = \frac{1}{R_i},$$

with $R_i$ radius of smallest/largest osculating circle.

# Principal directions



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# Second fundamental form

Deriving $\nu$ gives $\mathrm{d}\nu : TM \to TM$ (using translation). Define

$$\mathrm{II}(p)(v, w) = -\langle \mathrm{d}\nu(v), w \rangle.$$

Diagonalize II using principal directions

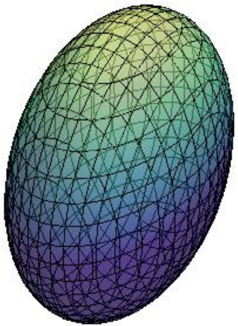
$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

## Theorema Egregium

The Gaussian curvature is independent of embedding.

# Approximating convex surfaces

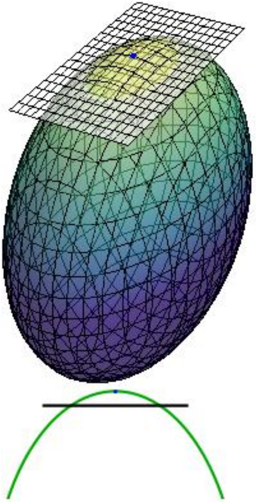
# Fejes Tóth: sketch for lower bound



Strictly convex surface: second fundamental form positive definite.

Principal curvatures $k_1, k_2 > 0$.

# Family of triangles

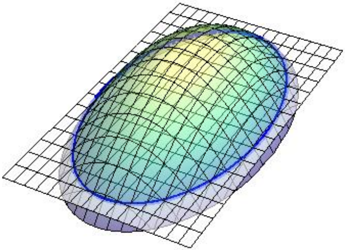


For any point we have a family of triangles:

- vertices on surface
- same one-sided Hausdorff distance to surface
- Hausdorff distance attained in given point

$\Rightarrow$ Each triangle lies in same plane

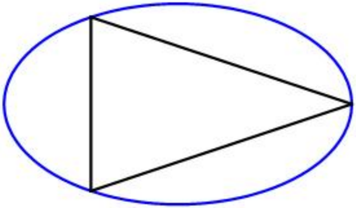
# Triangles inscribe an ellipse



Intersection approximately ellipse with semi-axis $\sqrt{2d_H R_1}$ and $\sqrt{2d_H R_2}$, with

$$R_i = \frac{1}{k_i}$$

$d_H$ (one-sided) Hausdorff distance

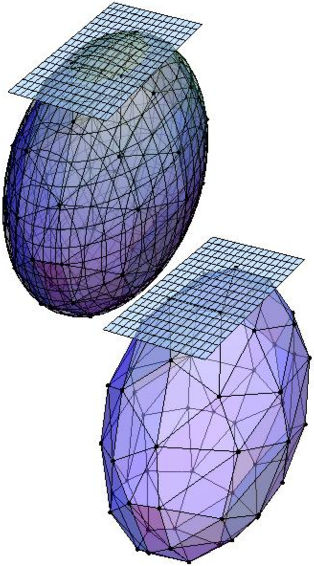


- Aim triangulation: few vertices ⇒ big triangles
- Inscribed triangle with greatest area

$$A = \frac{\sqrt{27}}{2}\sqrt{d_H^2 R_1 R_2} + o(d_H)$$

- for each triangle by inversion

$$d_H \gtrsim \frac{2}{\sqrt{27}}\sqrt{KA}$$



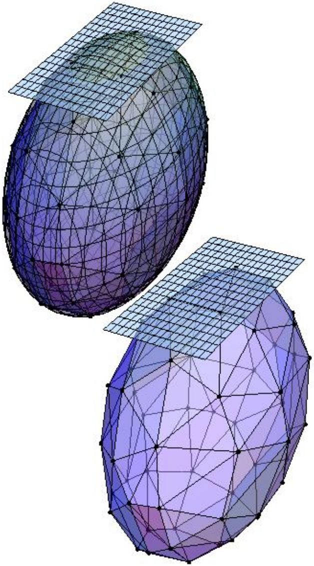
- for each triangle

$$d_H \gtrsim \frac{2}{\sqrt{27}}\sqrt{KA}$$

- Summing over all triangles

$$d_H n_\triangle \gtrsim \sum_{n_\triangle} \frac{2}{\sqrt{27}}\sqrt{K_\triangle A_\triangle}$$

- Area of triangle converges to the area of its projection along the normal



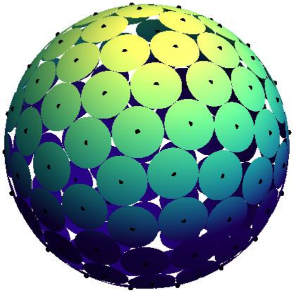
- number of triangles to infinity

$$\lim_{n_\triangle \to \infty} d_H n_\triangle \gtrsim \frac{2}{\sqrt{27}} \int \sqrt{K} \mathrm{d}A$$

- $n_\triangle \sim 2n$, with n number of vertices, yields

$$\lim_{n \to \infty} d_H n \gtrsim \frac{1}{\sqrt{27}} \int \sqrt{K} \mathrm{d}A$$

# Optimal packing



Optimal packing:

- Fixed number of points
- Maximize the minimal geodesic distance between points
- Geodesic balls of half the distance (optimal packing radius) do not intersect

Packing density converges to

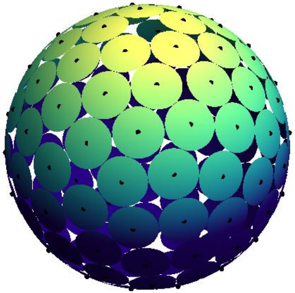
$$\frac{1}{\sqrt{27}}$$

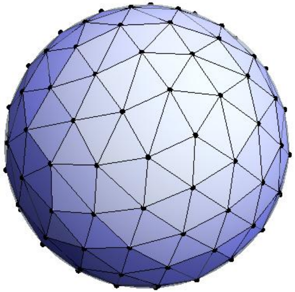
# Schneider: precise proof

Lower bound realization:

- Surface strictly convex so II positive definite diagonalizes to
$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$
$\Rightarrow$ metric
- Optimal packing of geodesic balls for new metric
- Take the centres
- Construct convex hull

# Sphere





Closest thing to a regular *n*-polytope for sphere

# Results

Convex Polytopes whose vertices lie on the surface:

$$\lim_{n\to\infty} d_H n = \frac{1}{\sqrt{27}} \int \sqrt{K} \mathrm{d}A$$

$$\lim_{n\to\infty} d_H n = \frac{2^{(1-n)/2}\theta_{n-1}}{k_{n-1}} \int \sqrt{K} \mathrm{d}A,$$

with $\theta_n$ packing density and $k_n$ volume $B^n$.
Convex Polytopes whose vertices lie in the ambient space:

$$\lim_{n\to\infty} d_H n = \frac{1}{2\sqrt{27}} \int \sqrt{K} \mathrm{d}A$$

Conjectured by Fejes Tóth (proof in progress report).

# Towards non-convex surfaces

# Clarkson: upper bound non-convex

- Use Taylor:

$$f(x) = f(p) + \nabla f(p)x + \frac{x^t H(p) x}{2} + \mathcal{O}(\|x\|^3)$$

- linear approximation

$$\text{Error}(f)(x) = \left| \frac{x^t H(p) x}{2} \right| + \mathcal{O}(\|x\|^3)$$

- Diagonalize the Hessian and take absolute values ⇒ ‘metric’
- Follow Schneider

# Lower bound

Can we employ local techniques?
Does length of edges tend to zero?
Obvious if no lines lie on surface:

- Family of optimal triangulations $T_n$

$$\lim_{n\to\infty} d_H(T_n, \Sigma) = 0$$

- convergent sequence of edges in triangulation $E_n \subset T_n$
- Assume

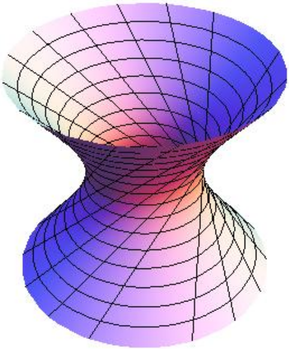
$$\lim_{n\to\infty} \text{length}(E_n) \neq 0$$

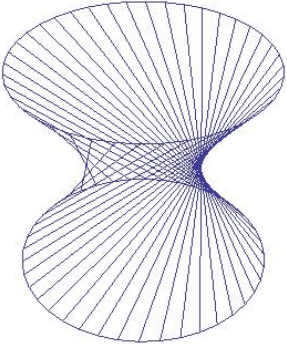
- Then

$$\lim_{n\to\infty} E_n = E \subset \Sigma$$

is a straight line

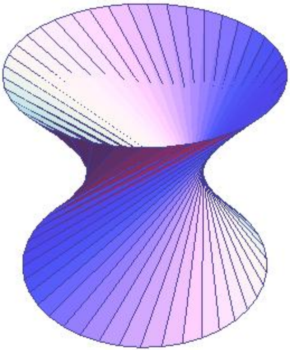
# Ruled Surfaces





Surfaces containing straight lines

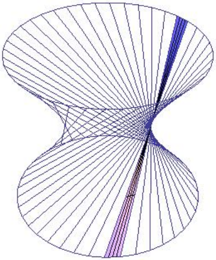
# Ruled Surfaces

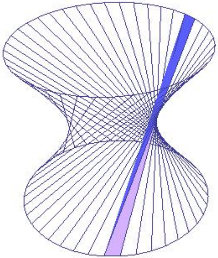


Fejes Tóth:

- Maximum distance attained near boundary
- Much better approximation

Wrong on both accounts





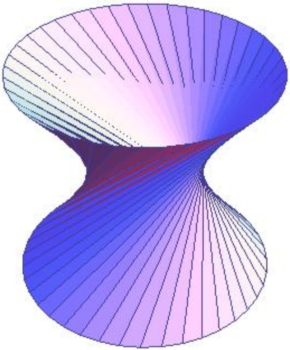
## Conjecture

The edge length of triangles in a triangulation tends to zero

$\Rightarrow$ Can employ local techniques

Clarkson: ‘roughly quadratic surfaces’

# Area comparison



Stronger than Schwarz Lantern:

- Good convergence

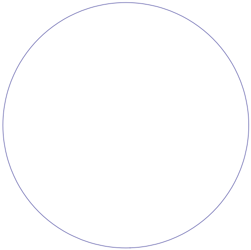
$$\lim_{n\to\infty} n\, d_H(T_n, \Sigma) = c$$

- Area gives problems

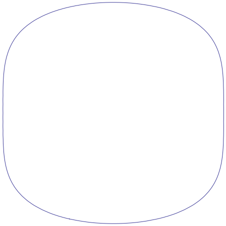
$$\lim_{n\to\infty} \text{area}(T_n) \neq \text{area}(\Sigma)$$

# Arbitrary codimension

Approximations by piecewise quadratic surfaces generally not intrinsic: the flat torus $S^1 \times S^1 \subset \mathbb{R}^4$

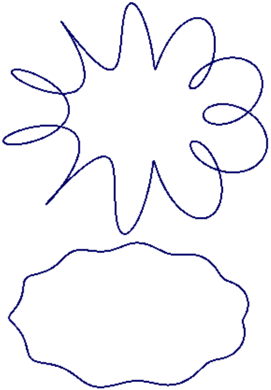


$x^2 + y^2 = 1$



$(x/c)^2 + (y/c)^4 = 1$

# Triangulations



Embed the flat torus in $\mathbb{R}^8$

$$\cos(\theta), \sin(\theta), \cos(m\theta)/m, \sin(m\theta)/m,$$
$$\cos(\varphi), \sin(\varphi), \cos(m\varphi)/m, \sin(m\varphi)/m$$

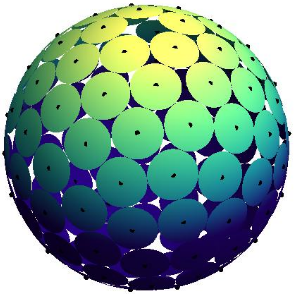
Approximate

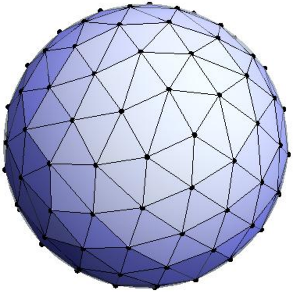
$$\lim_{n\to\infty} n d_H \geq c_m,$$

with $c_m \to \infty$ if $m \to \infty$
Implies there is no $\tilde{C}$, $f(g)$ such that

$$\lim_{n\to\infty} d_H n \leq \tilde{C} \int f(g) \mathrm{d}A$$

Extrinsic case: some upper bound by David de Laat





The pictures involving optimal packings were made using data from: "http://www2.research.att.com/~njas"

# The End