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Piecewise Linear Approximation of Contact Surfaces

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Outline

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Introduction

Contact Surfaces in Robot Motion Planning

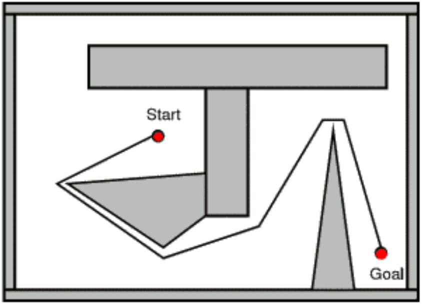
Piecewise Linear Approximation of Space Curves

Piecewise Linear Approximation of Cylindrical Surfaces

PL Approximation of Ruled Surfaces & Future Work

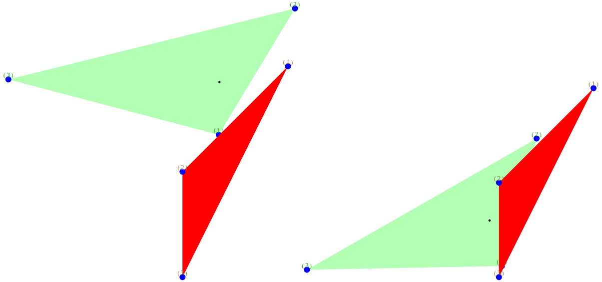
Motivation

- Study collision free motion of a robot from start configuration to goal configuration



Motivation

- Our Case : the robot and obstacle geometry is described in 2D, whereas the motion is represented in *configuration space* $\mathcal{C}$.
- Set of collision free configurations is called the free space $\mathcal{C}_{free}$ and $\mathcal{C}_{forb} := \mathcal{C} \setminus \mathcal{C}_{free}$ is forbidden space.



Problem Statement : I

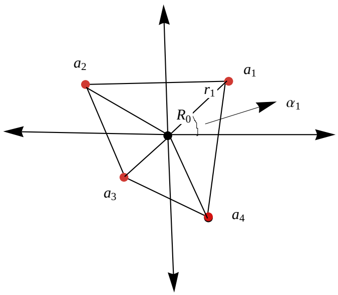
- *Contact Surfaces* : The configurations where the robot is in contact with an obstacle in a free manner generates curves and surfaces in $\mathcal{C}$.
- Goal : To find explicit parametrization of these contact surfaces.
- Result : Explicit parametrization of contact surfaces obtained. (To our knowledge : for the first time.)

Problem Statement : II

- Goal : Study and analyse the topology of the configuration space.
- Tools :
  - Piecewise linear approximation of curves of intersection of planes/surfaces with contact surfaces.
  - (Conservative) piecewise linear approximation of contact surfaces.

Definitions: Robot and Obstacle

- Given : convex polygon (robot) $\mathcal{A}$ with vertices $a_i = (a_{i,x}, a_{i,y}) = r_i(\cos\alpha_i, \sin\alpha_i)$



- Each obstacle $\mathcal{O}$ is also considered to be a convex polygon, where its vertices are $b_j = (b_{j,x}, b_{j,y})$.

Definitions: Configuration Space

- The configuration space $\mathcal{C}$ is given by:

$$\mathcal{C} = \{(x, y, \theta) | (x, y) \in \mathbb{R}^2,\ \theta \in [0, 2\pi)\}$$

- $\forall q \in \mathcal{C}$, $\mathcal{A}(q)$ gives a pose of the robot in the work space.
- $q = (x, y, \theta)$, where $\mathbf{r} = (x, y)$ and $a_i(q) = \mathbf{r} + R^{\theta} \cdot a_i$, here
- R^{θ} is rotation matrix in the plane.

Helices in Configuration Space

- Fixing a point a on $\partial\mathcal{A}$ and rotating the robot about this point,
- the reference point is parametrized by : $a + r_a(\cos\chi, \sin\chi)$, where $\chi \in [0, 2\pi)$.
- Locus of R_0 is a circle and the corresponding curve in the configuration space is a helix :

$$(a_x + r_a \cos\chi,\, a_y + r_a \sin\chi,\, \chi).$$

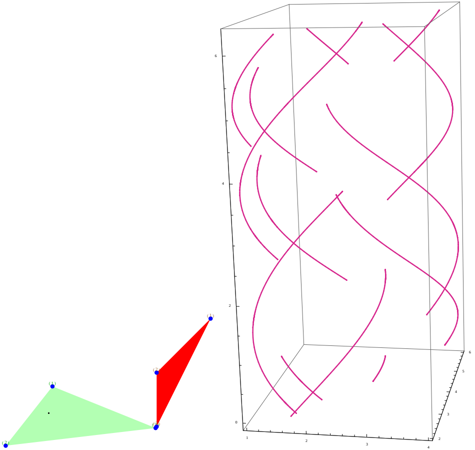
- In general, when a-th vertex coincides with a point P in workspace, we have

$$\mathbf{r}_a(\phi) = P - r_a\big(\cos(\phi + \alpha_a), \sin(\phi + \alpha_a)\big) \text{ and } \theta(\phi) = \phi \mod 2\pi.$$

- Note that helices form a cover of $\mathcal{C}$.

Contact Types : Vertex Vertex Contact

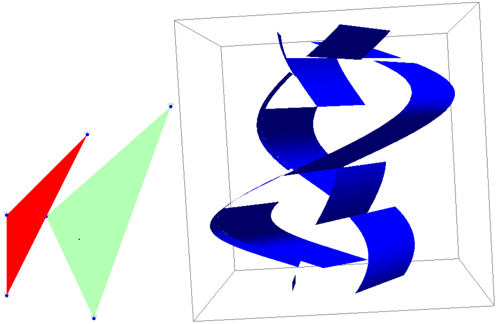
- Vertex of a robot in contact with a vertex of an obstacle yields helices in $\mathcal{C}$.



- $\left(b_{j,x} - r_i \cos(\phi + \alpha_i),\, b_{j,y} - r_i \sin(\phi + \alpha_i),\, \phi\right)$

Contact Types : Vertex Edge Contact

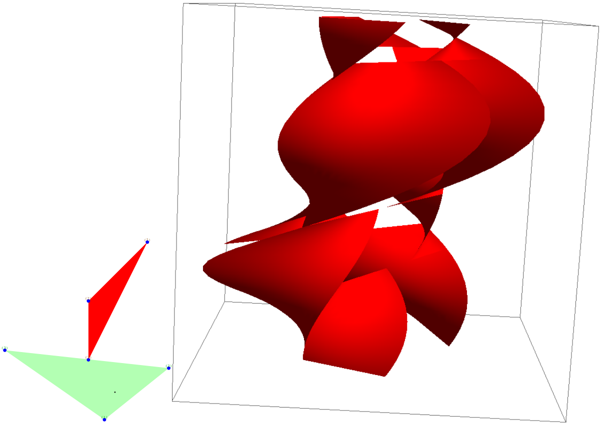
- Vertex of a robot in contact with an edge of an obstacle yields cylindrical surfaces, where the directrix is a helix.



- $\left(b_{j,x}+t\,(b_{j+1,x}-b_{j,x})-r_i\cos(\phi+\alpha_i),\ b_{j,y}+t\,(b_{j+1,y}-b_{j,y})-r_i\sin(\phi+\alpha_i),\ \phi\right)$

Contact Types : Edge Vertex Contact

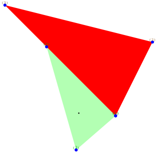
- Edge of a robot in contact with a vertex of an obstacle yields ruled surfaces.

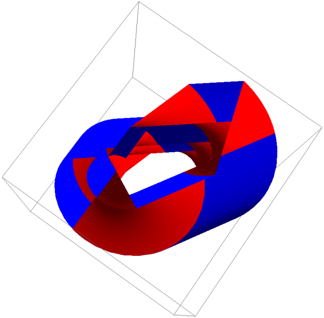


- $\left(b_{j,x} - a_{i,t,x} \cos\phi + a_{i,t,y} \sin\phi,\ b_{j,y} - a_{i,t,x} \sin\phi - a_{i,t,y} \cos\phi,\ \phi\right)$

Contact Types : Edge Edge Contact

- Edge of a robot in contact with an edge of an obstacle generates a line segment.





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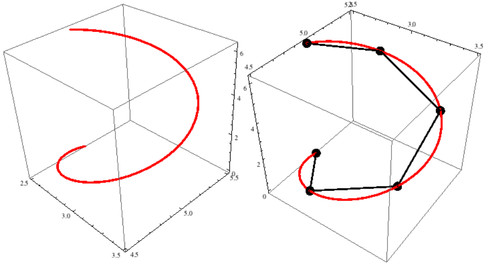
Outline

- Given : smooth (C^2 or better) curve α in space, $\varepsilon > 0$,
- Goal : approximate α with linear spline
  (i) within Hausdorff-distance ε
  (ii) of optimal complexity



Complexity Result : Linear Spline

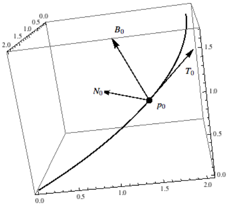
- **Féjes Toth, 1948** : For a convex C^2-curve in $\mathbb{R}^2$ the minimal number N_{min} of vertices of an inscribed, ε-approximating polygon is

$$N_{\min} = \frac{1}{2\sqrt{2}}\Big(\int_{s=0}^{L} \sqrt{\kappa(s)}\, ds\Big)\, \varepsilon^{-1/2} + O(1).$$

- We show that for a C^2-curve in $\mathbb{R}^3$ the minimal number N_{min} of vertices of an optimal linear spline is same as above.

Frenet Serret Frame -3D

- $\alpha : [0, L] \to \mathbb{R}^3$: arc length parametrization

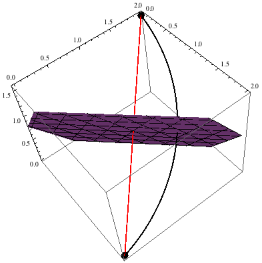


- Frenet-Serret :

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T \qquad -\tau B \\ B' &= \tau N \end{aligned}$$

Distance Function

- At every point on the curve we look at the normal plane and find its intersection point with the line segment.



- Local expression for Hausdorff distance that we obtained

$$\delta_H(\alpha, \beta) = \frac{1}{8} \kappa_0 \, \sigma^2 + O(\sigma^3).$$

Adaptive Bisection Algorithm

- Our algorithmic and theoretical results for complexity match exactly for various examples.
- Example : $(3 - \frac{1}{2}\cos t,\ 5 - \frac{1}{2}\sin t,\ t),\ t \in [0, 2\pi]$

ε	10^{-1}	2.5×10^{-2}	10^{-2}	5×10^{-3}	10^{-3}
Expt.	5	10	16	23	50
Theory	4.966	9.934	15.708	22.213	49.673

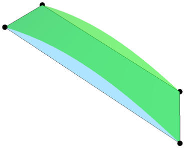
Outline

Definition of Cylindrical Surface

- $X : [0,\sigma] \times [0,\rho] \to \mathbb{R}^3$ sufficiently smooth cylindrical surface, parametrized by :

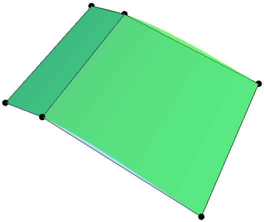
$$X(u,v) = \gamma(u) + v\,\mathbf{r}.$$

- $\gamma(u) = (x(u), y(u), z(u))$ is a space curve and $\mathbf{r} = (r_1, r_2, r_3)$, is a unit vector .
- Let $p_0 = X(0,0)$, $p_1 = X(\sigma,0)$, $p_2 = X(0,\rho)$ and $p_3 = X(\sigma,\rho)$.
- Note that above four points are coplanar and let $\Pi(p_0, p_1, p_2)$ denote this plane.



Problem Statement

- Given : sufficiently smooth cylindrical surface X, $\varepsilon > 0$,
- Goal: approximate X with linear spline
  (i) within one-sided Hausdorff-distance ε
  (ii) of optimal complexity



Complexity Result for PL Approximation

- We show that for a sufficiently smooth cylindrical surface in $\mathbb{R}^3$, the complexity of an optimal linear spline of an inscribed, ε-approximating polygon is

$$N(\varepsilon) = \frac{1}{2\sqrt{2}} \left(\int_0^L \sqrt{\frac{\kappa(u) \left((T(u) \times \mathbf{r}) \cdot N(u) \right)}{\|\mathbf{r} \times T(u)\|}}\, du \right) \varepsilon^{-1/2} + O(\varepsilon^{-1/3}).$$

- Here $\{T(u), N(u), B(u)\}$ is the Frenet-Serret frame of the space curve γ and u its arc length parameter.

Distance Function Definition

- The plane $\Pi(p_0, p_1, p_2) = a\,x + b\,y + c\,z + d$ is given by :

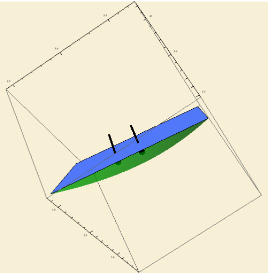
$$\begin{aligned} a &= r_3\,(y(0) - y(\sigma)) - r_2\,(z(0) - z(\sigma)) \\ b &= r_1\,(z(0) - z(\sigma)) - r_2\,(x(0) - x(\sigma)) \\ c &= r_2\,(x(0) - x(\sigma)) - r_1\,(y(0) - y(\sigma)) \text{ and} \\ d &= \mathbf{r} \times (\gamma(0) - \gamma(\sigma)). \end{aligned}$$

- For every point $X(u, v)$ we consider its orthogonal projection onto $\Pi(p_0, p_1, p_2)$.
- Normal to $\Pi(p_0, p_1, p_2)$ is $n_\Pi = (a, b, c)$
- The distance function dist(u) is therefore given by:

$$\mathrm{dist(u)} = \frac{-\langle \gamma(\mathrm{u}), \mathbf{r} \times (\gamma(0) - \gamma(\sigma)) \rangle + \langle \mathbf{r}, \gamma(0) \times \gamma(\sigma) \rangle}{\|(\gamma(0) - \gamma(\sigma)) \times \mathbf{r}\|}$$

One Sided Hausdorff Distance

- To compute the one sided Hausdorff distance, we need to find the maximum of this distance function in the interval $[0, \sigma]$



- Local expression for the one sided Hausdorff distance from X to Π is given by

$$\delta_H^O(X, \Pi) = \frac{1}{8} \frac{(T_0 \times \mathbf{r}) \cdot N_0}{\|\mathbf{r} \times T_0\|} \kappa_0 \, \sigma^2 + O(\sigma^3)$$

Adaptive Bisection Algorithm

- Subdivide along curve $\gamma(u)$ and measure distance between the given surface and the corresponding parallelogram.
- Our algorithmic and theoretical results for complexity match exactly for various examples.
- Example : $(3 - \frac{v}{\sqrt{2}} - 0.5 \cos u,\ 5 - \frac{v}{\sqrt{2}} - 0.5 \sin u, u),\ u \in [4.45689, 5.74277]$ and $v \in [0, \sqrt{2}]$.

ε	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
Expt.	3	10	30	93	294
Theory	2.930	9.267	29.3054	92.6718	293.054

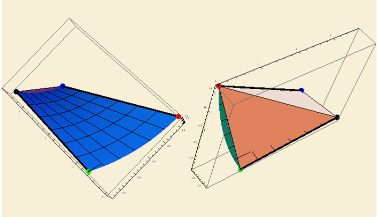
Outline

PL Approximation of Ruled Surfaces

- In the edge-vertex contact case, the surfaces we get are ruled, and in general ruled surfaces have the form :

$$S(u, v) = \gamma(u) + v\,\omega(u).$$

- Goal : To find a good way to sample points from this surface within a given Hausdorff distance ε.



PL Approximation of Ruled Surfaces

- In CGL technical report *On the Complexity of Polyhedral Approximations of Submanifolds in Euclidean Spaces*, M.H.M.J. Wintraecken and G. Vegter have considered the asymptotic setting of PL approximation of ruled surfaces.
- They claim for long triangles that

$$n * \delta_H(T, M) \geq \frac{1}{\sqrt{27}} \int_M \sqrt{|K|} dA,$$

Future Work

- Develop an algorithm to sample points from the ruled surface to obtain an optimal approximation.
- Consider conservative approximation by looking at offsets of piecewise linear surfaces.
- Consider various possibilites for reference point R_0 and see how it affects the contact surfaces.
- Study arrangements of curves on surfaces.

Thank You!