Stability of Delaunay-type structures for manifolds

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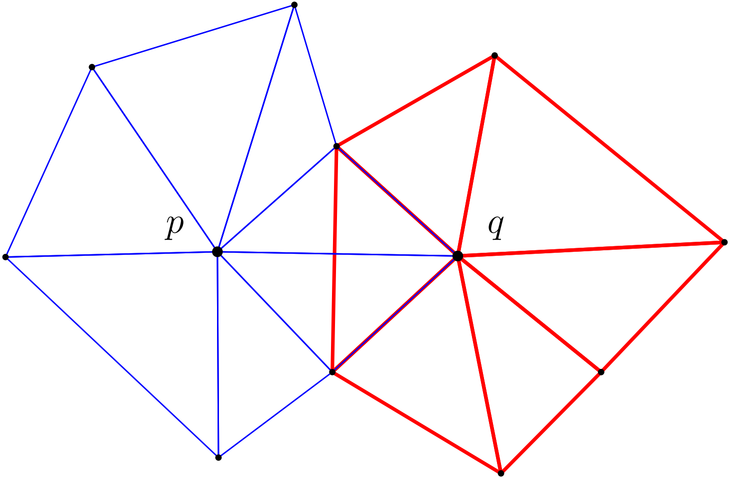
Manifold reconstruction

No canonical Delaunay structure

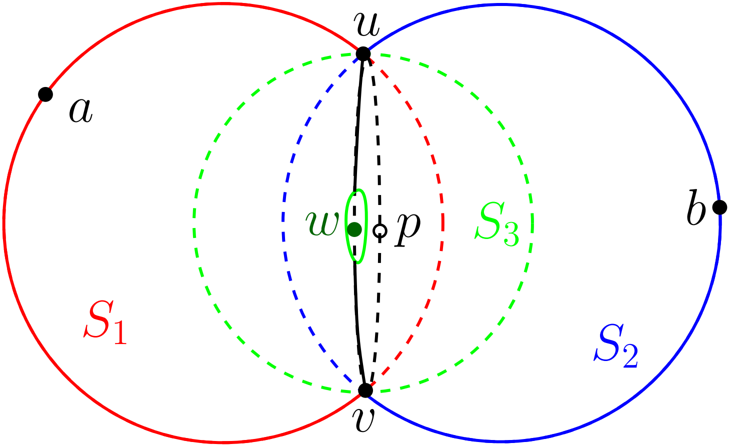
Local Delaunay patches

Coherence problems



Topological defects

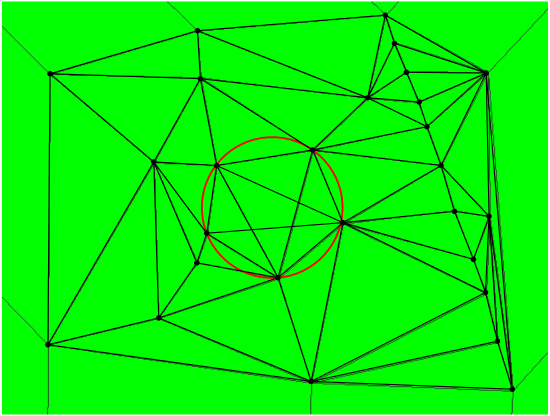
Intrinsic Delaunay triangulations?



Outline

Delaunay complexes

Establishing our terminology



- *all* simplices with an empty circumscribing ball
- only consider interior points

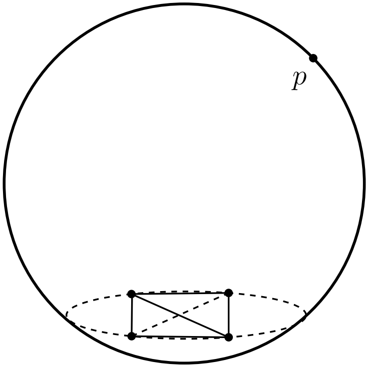
General position for Delaunay triangulations

Definition (General position (Delaunay 1934))

$\mathsf{P} \subset \mathbb{R}^m$ is in general position if there is no empty ball with more than $m+1$ points of P on its boundary.

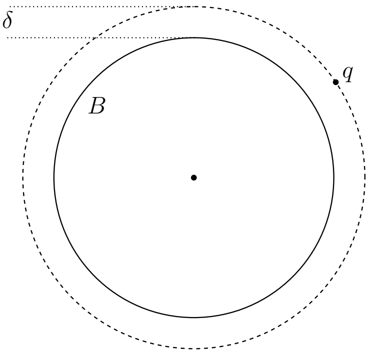
Degenerate Delaunay simplices

the ultimate slivers



- degenerate Delaunay j-simplex $\Longrightarrow$ $j+1$ points on a $(j-2)$-sphere
- *always* a face of a degenerate Delaunay $m+1$-simplex
- $\Longrightarrow$ $m+2$ points on the boundary of a Delaunay ball

Protection



Definition (protected)

A simplex σ is *protected* if it has a Delaunay ball B whose boundary contains no other points from P.
We say σ is δ-*protected* if $d_{\mathbb{R}^d}(q, \partial B) > \delta$ for all $q \in \mathsf{P} \setminus \sigma$.

Quantifying genericity

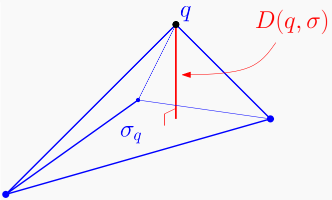
Definition (δ-generic)

A point set $\mathsf{P} \subset \mathbb{R}^m$ is δ-*generic* if the Delaunay m-simplices are all δ-protected.

- guarantees no degenerate m-simplices
- this definition is uninteresting without a sampling radius, ϵ
- $\nu_0^{-1} = \epsilon/\delta$ like a condition number

Towards a simplex quality bound

Altitudes



If σ_q, the face opposite q in σ is protected, then the *altitude* of q,

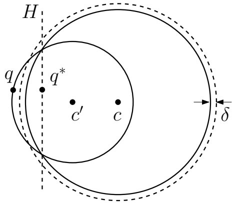
$$D(q,\sigma) = d_{\mathbb{R}^m}(q, \operatorname{aff}(\sigma_q)),$$

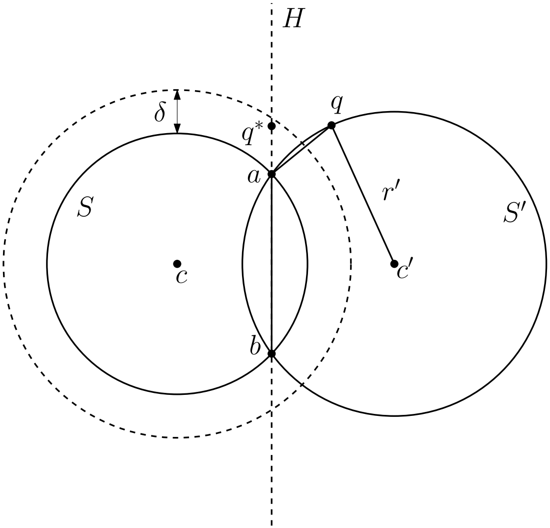
is bounded.

Assuming $\operatorname{aff}(\mathsf{P}) = \mathbb{R}^m$, and considering only interior simplices:

- If σ^j is a Delaunay j-simplex, with $j < m$, then for any $q \in \mathsf{P} \setminus \sigma^j$, there is a Delaunay m-simplex, σ^m, with $\sigma^j \leq \sigma^m$ and $q \notin \sigma^m$.
- $\implies$ a δ-generic point set is δ-sparse: $d_{\mathbb{R}^m}(p,q) > \delta$ for all $p, q \in \mathsf{P}$

Bounding the altitudes





Using $d_{\mathbb{R}^m}(q,a) > \delta$, and $d_{\mathbb{R}^m}(a,b) > \delta$, we bound $\angle qab$.

We get $D(q,\sigma) > \frac{\sqrt{3}}{2}\frac{\delta^2}{\epsilon}$.

Thickness

The *thickness* of a j-simplex σ with diameter $\Delta(\sigma)$ is

$$\Upsilon(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{p \in \sigma} \frac{D(p,\sigma)}{j\Delta(\sigma)} & \text{otherwise.} \end{cases}$$

Result

If P is δ-generic, with $\delta = \nu_0 \epsilon$, then

$$\Upsilon(\sigma) > \Upsilon_0 = \frac{\sqrt{3}\nu_0^2}{4},$$

for all deep interior Delaunay simplices σ.

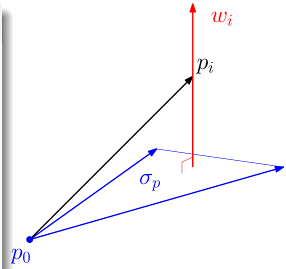
Thickness and singular values

Lemma

Let $\sigma = [p_0, \ldots, p_j]$, and let P be the $d \times j$ matrix whose columns are $p_i - p_0$. Then $s_1(P) \leq \sqrt{j}\Delta(\sigma)$, and

$$s_j(P) \geq \sqrt{j}\Upsilon(\sigma)\Delta(\sigma).$$

Thus $\Upsilon(\sigma)^{-1} \geq \frac{s_1(P)}{s_j(P)} = \kappa(P)$, the condition number of P.



Almost circumcentres

Definition

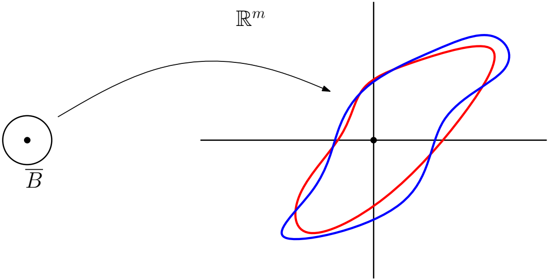
$\tilde{\rho}$-centre A point x is a $\tilde{\rho}$-centre for σ if $|d_{\mathbb{R}^m}(x,p) - d_{\mathbb{R}^m}(x,q)| \leq \tilde{\rho}$ for all vertices $p, q \in \sigma$.

Lemma

Suppose σ is an Υ_0-thick m-simplex with $L(\sigma) \geq \mu_0 \epsilon$. If x is a $\tilde{\rho}$-centre for σ with $d_{\mathbb{R}^m}(x,p) < 2\epsilon$ for all $p \in \sigma$, then

$$x \in B = B_{\mathbb{R}^m}(C(\sigma); \eta), \qquad \text{where } \eta = \frac{2\tilde{\rho}}{\Upsilon_0 \mu_0}.$$

Circumcentres under metric perturbation



$$f_e, f : \overline{B} \to \mathbb{R}^m$$

$$f_e : x \mapsto (d_{\mathbb{R}^m}(p_1, x) - d_{\mathbb{R}^m}(p_0, x), \dots, d_{\mathbb{R}^m}(p_m, x) - d_{\mathbb{R}^m}(p_0, x))^{\mathsf{T}}$$

$$f : x \mapsto (d(p_1, x) - d(p_0, x), \dots, d(p_m, x) - d(p_0, x))^{\mathsf{T}}$$

$f^{-1}(0) \neq \emptyset \implies$ there is in B a circumcentre for σ w.r.t. the the metric d

Circumcentres under metric perturbation

Lemma

Let $U \subset \mathbb{R}^m$, and let $d : U \times U \to \mathbb{R}$ be a metric topologically equivalent to $d_{\mathbb{R}^m}$ and such that for any $x, y \in U$ with $d_{\mathbb{R}^m}(x, y) < 2\epsilon$, we have $|d(x, y) - d_{\mathbb{R}^m}(x, y)| \leq \rho$, with

$$\rho \leq \frac{\Upsilon_0 \mu_0 \epsilon}{4\sqrt{m}}.$$

If $\sigma = [p_0, \ldots, p_m] \subset U$ is an Υ_0-thick m-simplex with $R(\sigma) < \epsilon$, and $L(\sigma) \geq \mu_0 \epsilon$, and such that $d_{\mathbb{R}^m}(p_i, \partial U) \geq 2\epsilon$, then there is a point

$$c \in B = B_{\mathbb{R}^m}(C(\sigma); \frac{4\sqrt{m}\rho}{\Upsilon_0 \mu_0})$$

such that $d(c, p_i) = d(c, p_j)$ for all $p_i, p_j \in \sigma$.

Circumcentres under metric perturbation

Lemma

Let $U \subset \mathbb{R}^m$*, and let* $d : U \times U \to \mathbb{R}$ *be a metric topologically equivalent to* $d_{\mathbb{R}^m}$ *and such that for any* $x, y \in U$ *with* $d_{\mathbb{R}^m}(x,y) < 2\epsilon$*, we have* $|d(x,y) - d_{\mathbb{R}^m}(x,y)| \leq \rho$*, with*

$$\rho \leq \frac{\Upsilon_0 \mu_0 \epsilon}{4\sqrt{m}}.$$

If $\sigma = [p_0, \ldots, p_m] \subset U$ *is an* Υ_0*-thick* m*-simplex with* $R(\sigma) < \epsilon$*, and* $L(\sigma) \geq \mu_0 \epsilon$*, and such that* $d_{\mathbb{R}^m}(p_i, \partial U) \geq 2\epsilon$*, then there is a point*

$$c \in B = B_{\mathbb{R}^m}(C(\sigma); \frac{4\sqrt{m}\rho}{\Upsilon_0 \mu_0})$$

such that $d(c, p_i) = d(c, p_j)$ *for all* $p_i, p_j \in \sigma$.

Stability under metric perturbation

Theorem

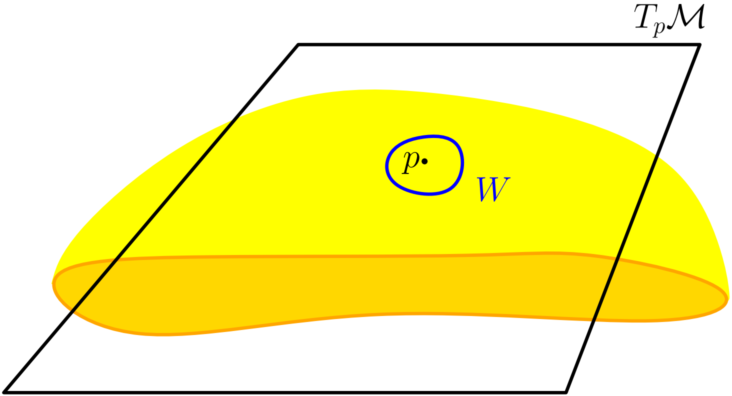
Suppose P *is* δ*-generic for* P_I*, with sampling radius* ϵ *and* $\delta = \nu_0 \epsilon$*. Suppose also that* $\mathrm{conv}(\mathsf{P}) \subseteq U$*, and* $d : U \times U \to \mathbb{R}$ *is such that* $|d(x,y) - d_{\mathbb{R}^m}(x,y)| \leq \rho$ *for all* $x, y \in U$*. If*

$$\rho \leq \frac{\nu_0^3}{48\sqrt{m}} \delta = \frac{\nu_0^4}{48\sqrt{m}} \epsilon,$$

then

$$\mathrm{star}(\mathsf{P}_I; \mathrm{Del}_d(\mathsf{P})) = \mathrm{star}(\mathsf{P}_I; \mathrm{Del}(\mathsf{P})).$$

Local Euclidean metrics



Locally project into $T_p\mathcal{M}$, which is identified with $\mathbb{R}^m$

$$\psi_p : \mathcal{M} \supset W \xrightarrow{\cong} U \subset T_p\mathcal{M}$$

$d_{\mathbb{R}^m}(\psi_p(x), \psi_p(y))$, a Euclidean metric on W

Theorem

Suppose $\tilde{\mathsf{P}} \subset \mathcal{M}$ *is* $(\tilde{\mu}_0\epsilon)$*-sparse with respect to* $d_{\mathbb{R}^N}$*, and every* m*-simplex* $\tilde{\sigma} \in \mathrm{Del}_{T\mathcal{M}}(\tilde{\mathsf{P}})$ *is* $\tilde{\Upsilon}_0$*-thick, and has, for every vertex* $p \in \tilde{\sigma}$*, a* $\check{\delta}^2$*-power-protected empty ball of radius less than* ϵ *centred on* $T_p\mathcal{M}$*, with* $\check{\delta} \geq \delta_0\tilde{\mu}_0\epsilon$*. If*

$$\epsilon \leq \frac{\tilde{\Upsilon}_0\tilde{\mu}_0^3\delta_0^2 \mathit{rch}(\mathcal{M})}{8 \times 10^4\sqrt{m}},$$

then

$$\mathrm{Del}_{T\mathcal{M}}(\tilde{\mathsf{P}}) = \mathrm{Del}_{\mathbb{R}^N|_{\mathcal{M}}}(\tilde{\mathsf{P}}) = \mathrm{Del}_{\mathcal{M}}(\tilde{\mathsf{P}}),$$

and for ϵ *sufficiently small, these will be homeomorphic to* $\mathcal{M}$*:*

$$\left|\mathrm{Del}_{\mathcal{M}}(\tilde{\mathsf{P}})\right| \cong \mathcal{M}.$$

Summary

Protection parameterizes general position

- bounded thickness for Delaunay simplices on δ-generic P
- quantified robustness with respect to perturbations
- sufficient conditions for the intrinsic Delaunay complex to be a triangulation (result currently relies on an embedding in $\mathbb{R}^N$)

Thank You.