Vertex enumeration for polytopes defined by oracles

Vissarion Fisikopoulos

Joint work with I.Z. Emiris (UoA), B. Gärtner (ETHZ)

Dept. of Informatics & Telecommunications, University of Athens

CGL final workshop, 02.Oct.2013

Outline

Polytopes & Oracles

Vertex enumeration in the oracle model

- Total polynomial time algorithm / known edge directions
- Beneath-and-Beyond based algorithm / triangulation producing
- Face-lattice producing algorithm

Outline

Polytopes & Oracles

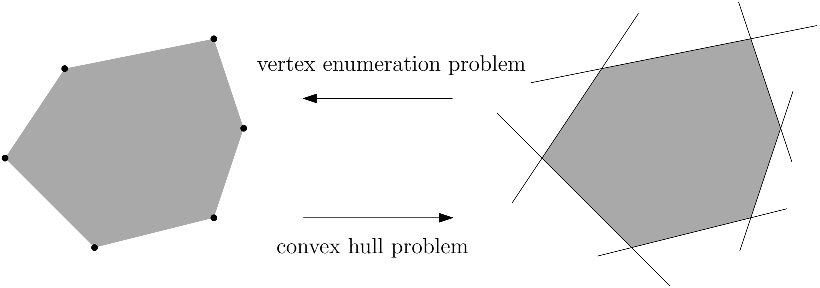
Vertex enumeration in the oracle model

- Total polynomial time algorithm / known edge directions
- Beneath-and-Beyond based algorithm / triangulation producing
- Face-lattice producing algorithm

Classical Polytope Representations

A convex polytope $P \subseteq \mathbb{R}^d$ can be represented as the

1. convex hull of a pointset $\{p_1, \ldots, p_n\}$ (V-representation)
2. intersection of halfspaces $\{h_1, \ldots, h_m\}$ (H-representation)



- These problems are equivalent by polytope duality.

Algorithmic Issues

- For general dimension d there is no polynomial algorithm for the convex hull (or vertex enumeration) problem since m can be $O\left(n^{\lfloor d/2 \rfloor}\right)$ [McMullen'70].

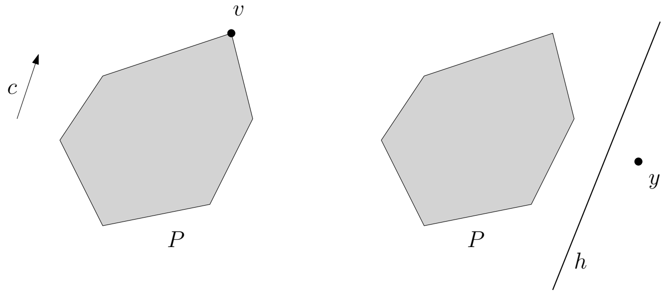
- It is open whether there exist a total poly-time algorithm for the convex hull (or vertex enumeration) problem, *i.e. runs in poly-time in* n, m, d.

Polytope Oracles

Implicit representation for a polytope $P \subseteq \mathbb{R}^d$.

OPT_P: Given direction $c \in \mathbb{R}^d$ return the vertex $v \in P$ that maximizes $c^T v$.

SEP_P: Given point $y \in \mathbb{R}^d$, return yes if $y \in P$ otherwise a hyperplane h that separates y from P.



Well-described polytopes and oracles

Definition

A rational polytope $P \subseteq \mathbb{R}^d$ is well-described (with a parameter φ) if there exists an H-representation for P in which every inequality has encoding length at most φ. The encoding length of P is $\langle P \rangle = d + \varphi$.

Proposition (Grötschel et al.'93)

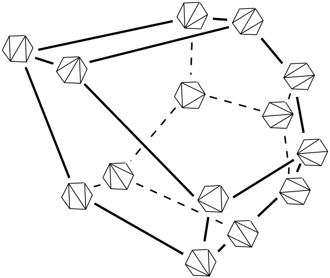
For a well-described polytope, we can compute OPT_P from SEP_P (and vice versa) in oracle polynomial-time. The runtime (polynomially) depends on d and φ.

Why oracles?

- Polynomial time algorithms for combinatorial optimization problems using the ellipsoid method [Grötschel-Lovász-Schrijver'93]
- Randomized polynomial-time algorithm for approximating the volume of convex bodies [Dyer-Frieze-Kannan '90]

Original Motivation

(1) Secondary, Resultant polytopes



- Vertices $\rightarrow$ reg. triangulations of a pointset's convex hull
- $\mathrm{OPT_P}$ is available via a triangulation computation
  [Emiris-F-Konaxis-Peñaranda '12]

(2) Minkowski sums

- Applications in Computational Algebraic Geometry, Geometric Modelling, Combinatorics

Outline

Polytopes & Oracles

Vertex enumeration in the oracle model

- Total polynomial time algorithm / known edge directions
- Beneath-and-Beyond based algorithm / triangulation producing
- Face-lattice producing algorithm

Main problems

Vertex enumeration in the oracle model

Given OPT_P for $P \subseteq \mathbb{R}^d$, compute the vertices of P.

Vertex enumeration in the oracle model with edge-directions

Given OPT_P and a superset D of the edge directions $D(P)$ of $P \subseteq \mathbb{R}^d$, compute the vertices of P.

Outline

Polytopes & Oracles

Vertex enumeration in the oracle model

- Total polynomial time algorithm / known edge directions
- Beneath-and-Beyond based algorithm / triangulation producing
- Face-lattice producing algorithm

Vertex enumeration with edge-directions

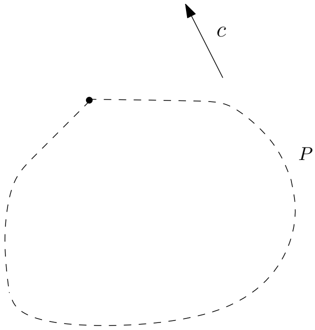
Given $\mathrm{OPT_P}$ and a superset D of the edge directions $\mathrm{D(P)}$ of $\mathrm{P} \subseteq \mathbb{R}^{\mathrm{d}}$, compute the vertices P.

Proposition (Rothblum-Onn '07)

Let $\mathrm{P} \subseteq \mathbb{R}^{\mathrm{d}}$ *given by* $\mathrm{OPT_P}$, *and* $\mathrm{D} \supseteq \mathrm{D(P)}$. *All vertices of* P *can be computed in*

$\mathrm{O}(|\mathrm{D}|^{\mathrm{d}-1})$ *calls to* $\mathrm{OPT_P} + \mathrm{O}(|\mathrm{D}|^{\mathrm{d}-1})$ *arithmetic operations.*

The edge-skeleton algorithm



Input:

- OPT_P
- Edge vec. P (dir. & len.): D

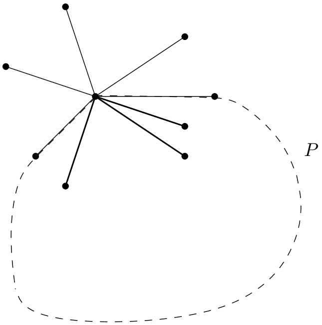
Output:

- Edge-skeleton of P

Sketch of **Algorithm**:

- Compute a vertex of P ($x = \mathrm{OPT}_P(c)$ for arbitrary $c^T \in \mathbb{R}^d$)

The edge-skeleton algorithm



Input:

- OPT_P
- Edge vec. P (dir. & len.): D

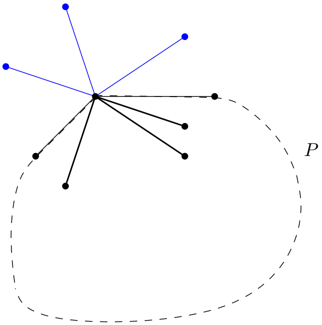
Output:

- Edge-skeleton of P

Sketch of **Algorithm**:

- Compute a vertex of P ($x = \mathrm{OPT}_P(c)$ for arbitrary $c^T \in \mathbb{R}^d$)
- Compute segments $S = \{(x, x + d)$, for all $d \in D\}$

The edge-skeleton algorithm



Input:

- OPT_P
- Edge vec. P (dir. & len.): D

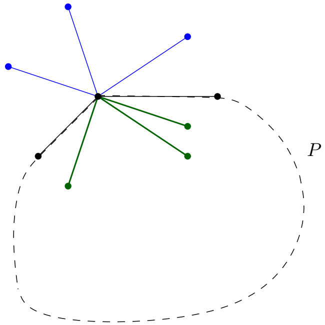
Output:

- Edge-skeleton of P

Sketch of **Algorithm**:

- Compute a vertex of P ($x = \mathrm{OPT}_P(c)$ for arbitrary $c^T \in \mathbb{R}^d$)
- Compute segments $S = \{(x, x+d)$, for all $d \in D\}$
- Remove from S all segments (x, y) s.t. $y \notin P$ ($\mathrm{OPT}_P \rightarrow \mathrm{SEP}_P$)

The edge-skeleton algorithm



Input:

- OPT_P
- Edge vec. P (dir. & len.): D

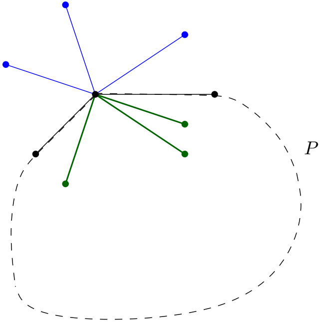
Output:

- Edge-skeleton of P

Sketch of **Algorithm**:

- Compute a vertex of P ($x = \mathrm{OPT}_P(c)$ for arbitrary $c^T \in \mathbb{R}^d$)
- Compute segments $S = \{(x, x + d)$, for all $d \in D\}$
- Remove from S all segments (x, y) s.t. $y \notin P$ ($\mathrm{OPT}_P \to \mathrm{SEP}_P$)
- Remove from S the segments that are not extreme

The edge-skeleton algorithm



Input:

- OPT_P
- Edge vec. P (dir. & len.): D

Output:

- Edge-skeleton of P

Sketch of **Algorithm**:

- Compute a vertex of P ($x = \mathrm{OPT}_P(c)$ for arbitrary $c^T \in \mathbb{R}^d$)
- Compute segments $S = \{(x, x+d)$, for all $d \in D\}$
- Remove from S all segments (x, y) s.t. $y \notin P$ ($\mathrm{OPT}_P \rightarrow \mathrm{SEP}_P$)
- Remove from S the segments that are not extreme

Can be altered to work with edge directions only

Complexity

Theorem

Given OPT_P and a superset of edge directions D of a well-described polytope $P \subseteq \mathbb{R}^d$, the edge skeleton of P can be computed in oracle total polynomial-time

$$O\left(n\ |D|\ T + n\ \mathbb{LP}(d^3|D|\langle B\rangle)\right),$$

- n *the number of vertices of* P,
- T : *runtime of oracle conversion algorithm for* P *and* D,
- $\langle B\rangle$ *is the binary encoding length of the vector set* D *and* P,
- $\mathbb{LP}(\langle A\rangle + \langle b\rangle + \langle c\rangle)$ *runtime of max* $c^T x$ *over* $\{x : Ax \leq b\}$.

Applications

Corollary

The edge skeleton of resultant, secondary polytopes can be computed in oracle total polynomial-time.

Corollary

The edge skeletons of polytopes appearing in convex combinatorial optimization [Rothblum-Onn '04] and convex integer programming [De Loera et al. '09] problems can be computed in oracle total polynomial-time.

Outline

Polytopes & Oracles

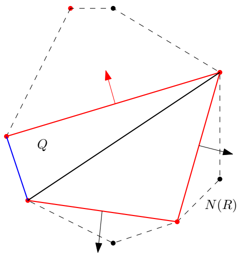
Vertex enumeration in the oracle model

- Total polynomial time algorithm / known edge directions
- Beneath-and-Beyond based algorithm / triangulation producing
- Face-lattice producing algorithm

Vertex enumeration in the oracle model

Beneath-and-Beyond based Algorithm (triangulation producing)

- [Emiris-F-Konaxis-Peñaranda '12] for resultant polytopes
- first: compute conv.hull of $d+1$ aff. indep. vertices of P
- step: call OPT_P with outer normal vector of a halfspace
  $\rightarrow$ either validate this halfspace
  $\rightarrow$ or add a new vertex to the convex hull



Vertex enumeration in the oracle model

Beneath-and-Beyond based Algorithm (triangulation producing)

- [Emiris-F-Konaxis-Peñaranda '12] for resultant polytopes
- first: compute conv.hull of $d+1$ aff. indep. vertices of P
- step: call OPT_P with outer normal vector of a halfspace
  $\rightarrow$ either validate this halfspace
  $\rightarrow$ or add a new vertex to the convex hull

Complexity

Given $P \subseteq \mathbb{R}^d$, H-, V-repr. & triang. T of P can be computed in

$$O(d^5 n s^2) \text{ arithmetic operations } + O(n+m) \text{ calls to } \mathrm{OPT}_P$$

s is the number of cells of T.

Vertex enumeration in the oracle model

Beneath-and-Beyond based Algorithm (triangulation producing)

- [Emiris-F-Konaxis-Peñaranda '12] for resultant polytopes
- first: compute conv.hull of $d+1$ aff. indep. vertices of P
- step: call OPT_P with outer normal vector of a halfspace
  $\rightarrow$ either validate this halfspace
  $\rightarrow$ or add a new vertex to the convex hull

Complexity

Given $P \subseteq \mathbb{R}^d$, H-, V-repr. & triang. T of P can be computed in

$$O(d^5 n s^2) \text{ arithmetic operations } + O(n+m) \text{ calls to } \mathrm{OPT}_P$$

s is the number of cells of T.

BUT: s can be $O\left(n^{\lfloor d/2 \rfloor}\right)$

Outline

Polytopes & Oracles

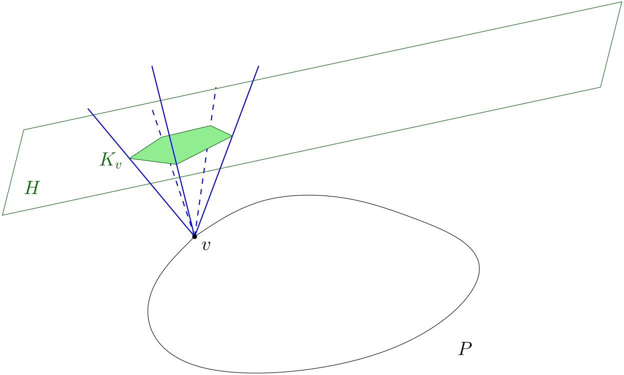
Vertex enumeration in the oracle model

- Total polynomial time algorithm / known edge directions
- Beneath-and-Beyond based algorithm / triangulation producing
- Face-lattice producing algorithm

Vertex enumeration in the oracle model

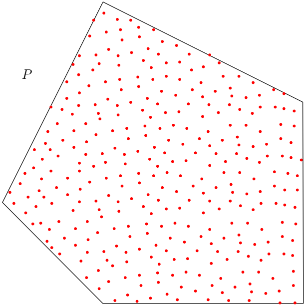
Recursive algorithm (face-lattice producing)

- For $P \subseteq \mathbb{R}^d$ define K_v the intersection of normal cone of vertex v with hyperplane H s.t. $\dim(K_v) = d - 1$.
- Given vertex $v \in P \subseteq \mathbb{R}^d$ and OPT_P compute in oracle polynomial time OPT_{K_v} for $K_v \subseteq \mathbb{R}^{d-1}$.



Polytope Volume Approximation

- Volume approximation of P reduces to uniform sampling from P [Dyer et.al '91]
- Random walks on convex bodies (e.g. hit-&-run)
- Assume membership (weak case of SEP) oracle



Ongoing work

- Study case of OPT oracle
- Implementation of both cases (MEM and OPT oracles)

Thank you!